

ECE 463/663 - Homework #5

Pole Placement.

Pole Placement (Inverted Pendulum)

Problem 1) (30pt) Use the dynamics of a Cart and Pendulum System from homework set #4:

$$s \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -29.4 & 0 & 0 \\ 0 & 26.133 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -0.667 \end{bmatrix} F$$

Design a feedback control law of the form

$$U = K_r * R - K_x * X$$

so that the closed-loop system has

- A 2% settling time of 10 seconds, and
- 5% overshoot for a step input

Start with where the closed-loop dominant pole belongs

$$s = -0.4 + jX \quad T_s = 10 \text{ seconds}$$

$$\zeta = 0.6898 \quad 5\% \text{ overshoot}$$

$$s = -0.4 + j0.4198$$

Let's round to

$$s = -0.4 + j0.42$$

Place the other two poles at -2 and -3 (somewhat arbitrary). Find K_x and K_r using pole placement

```
>> A = [0,0,1,0;0,0,0,1;0,-29.4,0,0;0,26.133,0,0]
```

```

0      0      1.0000      0
0      0      0      1.0000
0  -29.4000      0      0
0   26.1330      0      0

```

```
>> B = [0;0;1;-0.667]
```

```

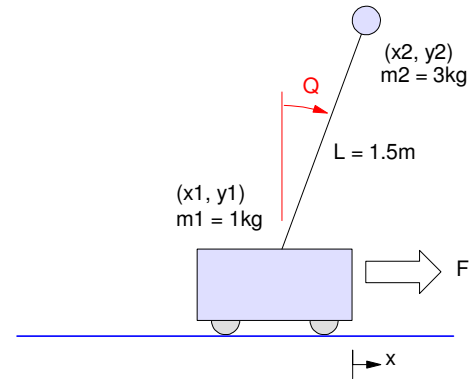
0
0
1.0000
-0.6670

```

```
>> C = [1,0,0,0]
```

```
C =      1      0      0      0
```

```
>> Kx = ppl(A, B, [-0.4+0.4j,-0.4-0.4j,-2,-3])
```



```
Kx =    -0.2943   -55.0935    -0.9811   -10.1666
```

```
>> DC = -C*inv(A-B*Kx)*B
```

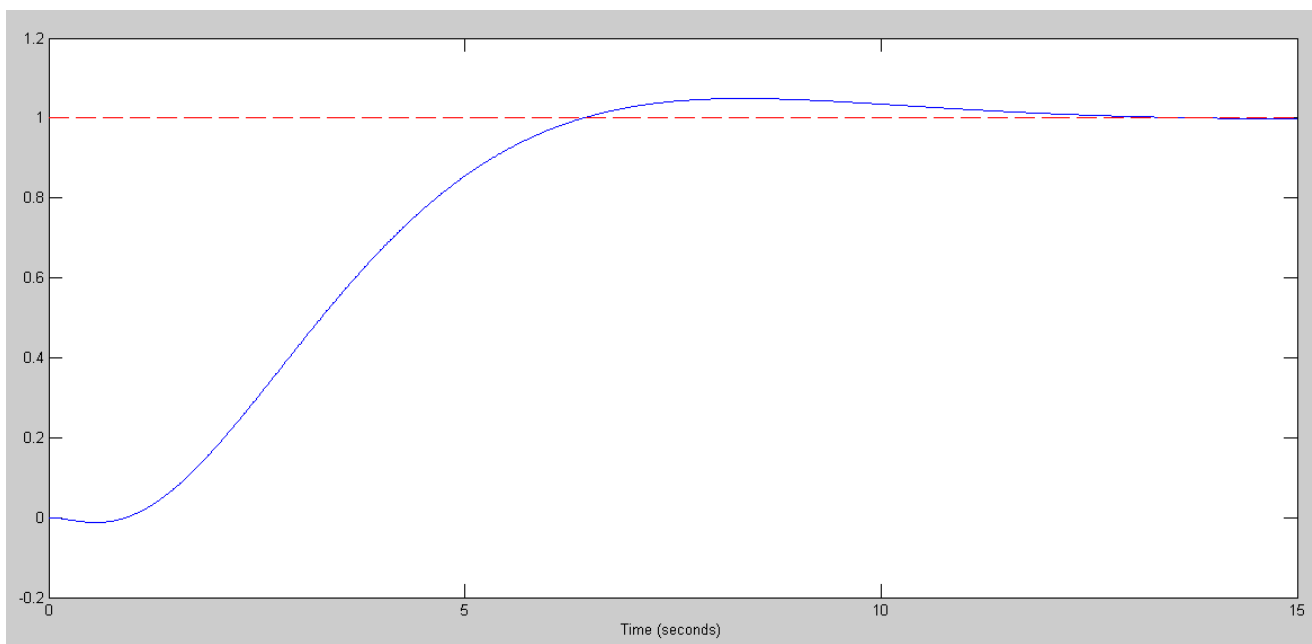
```
DC =    -3.2319
```

```
>> Kr = 1/DC
```

```
Kr =    -0.3094
```

Check the step response of the linearized system

```
>> G = ss(A - B*Kx, B*Kr, C, 0);  
>> t = [0:0.01:15]';  
>> y = step(G,t);  
>> Ref = 0*t + 1;  
>> plot(t,y,'b',t,Ref,'r--')  
>> xlabel('Time (seconds)');
```



Looks reasonable.

Try it on the nonlinear system. Change Cart_Dynamics to have the correct mass and lengths

```
function [ dX ] = CartDynamics( X, F )
%cart dynamics
m1 = 1; % cart
m2 = 3; % ball
L = 1.5; % length

% X = [x, q, dx, dq]
x = X(1);
q = X(2);
dx = X(3);
dq = X(4);
g = 9.8;
M = [m1+m2, m2*L*cos(q); m2*L*cos(q), m2*L*L];
A = [dq*dq*m2*L*sin(q); g*L*m2*sin(q)];
B = [1;0];
d2X = inv(M) * (A + B*F);
dX = [dx; dq; d2X];
end
```

Add the control law to the nonlinear simulation

```
% Cart and Pendulum

X = [0,0,0,0]';
Ref = 1;
dt = 0.01;
t = 0;
n = 0;
y = [];

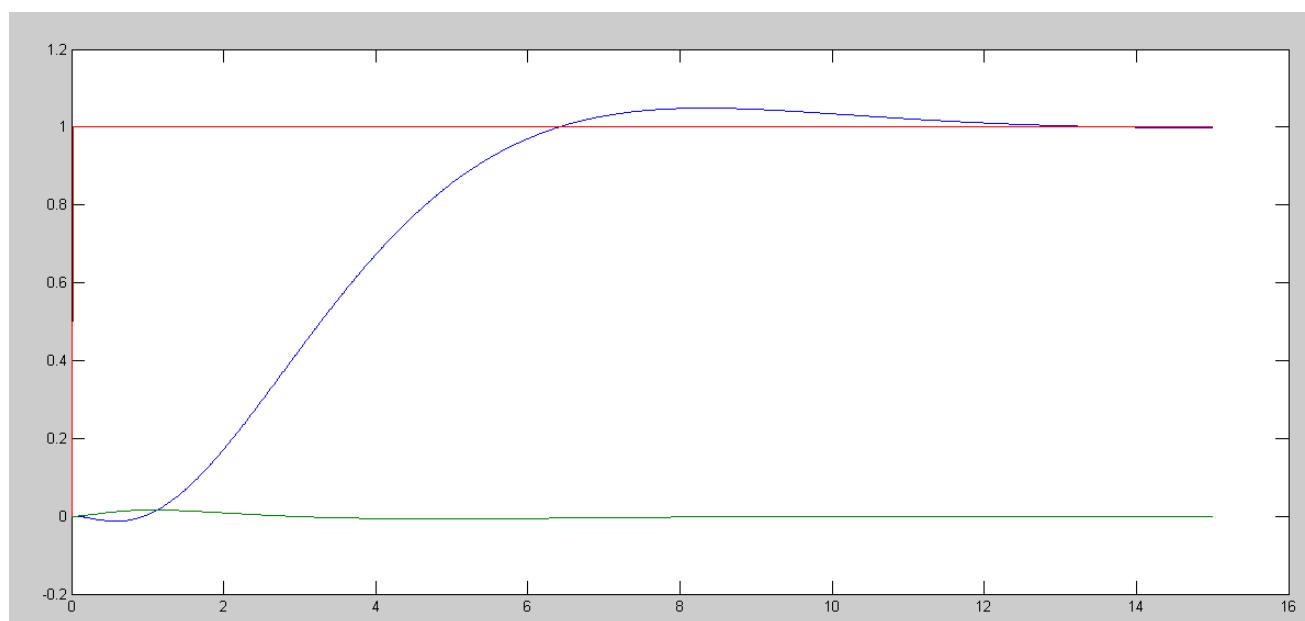
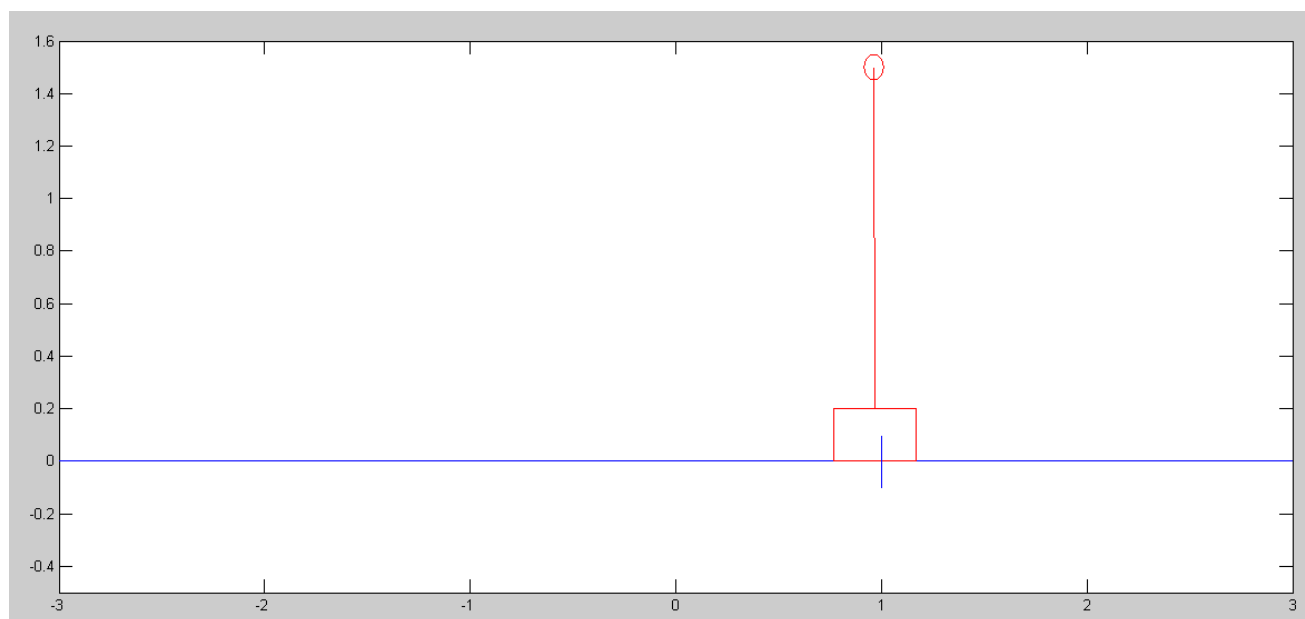
Kx = [-0.3094 -55.1407 -0.9937 -10.1854];
Kr = -0.3094;

while(t < 15)
    Ref = 1*(sin(0.2*t) > 0);
    U = Kr*Ref - Kx*X;
    dX = CartDynamics(X, U);
    X = X + dX * dt;
    t = t + dt;
    n = mod(n+1, 5);
    if(n == 0)
        CartDisplay(X, Ref);
    end
    y = [y ; X(1), X(2), Ref];
end

t = [1:length(y)]' * dt;
plot(t,y);
```

Let her rip

- The response is about what the linear system predicts
-



Pole Placement (Ball & Beam)

Problem 2) (30pt) Use the dynamics for the Ball and Beam system from homework set #4.

$$s \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -7.84 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \end{bmatrix} T$$

Design a feedback control law of the form

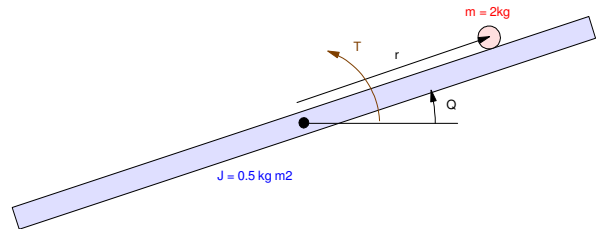
$$U = K_r * R - K_x * X$$

so that the closed-loop system has

- A 2% settling time of 10 seconds, and
- 5% overshoot for a step input

Check the step response of the linear system in Matlab

(Check the step response of the nonlinear system



Pretty much follow the same procedure for problem #1, only with a different A and B matrix. Place the closed-loop dominant poles at

$$s = -0.4 + j0.42$$

Place the other poles at -2 and -3 (somewhat arbitrary). In Matlab

```
>> A = [0,0,1,0;0,0,0,1;0,-7,0,0;-7.84,0,0,0]
```

```

      0      0      1.0000      0
      0      0      0      1.0000
      0     -7.0000      0      0
    -7.8400      0      0      0

```

```
>> B = [0;0;0;0.4]
```

```

      0
      0
      0
    0.4000

```

```
>> C = [1,0,0,0]
```

```

      1      0      0      0

```

```
>> Kx = ppl(A, B, [-0.4+0.42j,-0.4-0.42j,-2,-3])
```

```
Kx = -20.3209    25.8410    -2.3150    14.5000
```

```
>> DC = -C*inv(A-B*Kx)*B
```

```
DC = -1.3872
```

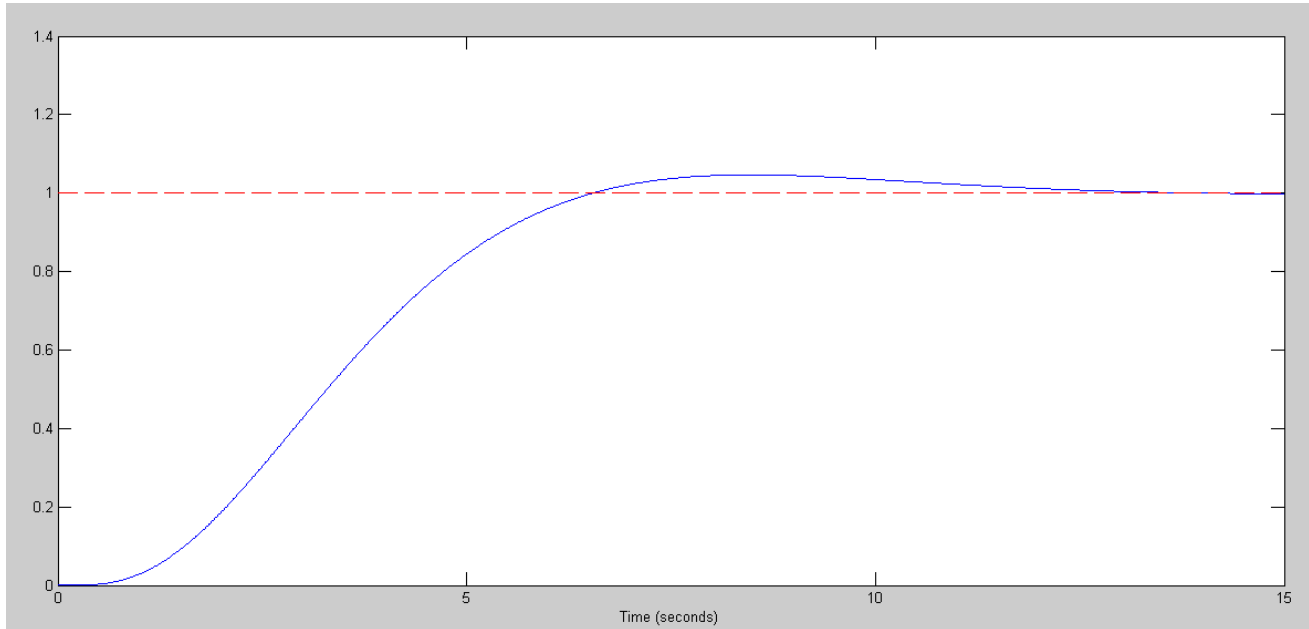
```
>> Kr = 1/DC
```

```
Kr = -0.7209
```

```
>>
```

Check the step response of the linear model

```
>> G = ss(A - B*Kx, B*Kr, C, 0);  
>> t = [0:0.01:15]';  
>> y = step(G,t);  
>> Ref = 0*t + 1;  
>> plot(t,y,'b',t,Ref,'r--')  
>> xlabel('Time (seconds)');
```



Looks reasonable. Try the nonlinear model. First, change the dynamics routine to match the system

```
function [dX] = BeamDynamics( X, T )
```

```
m = 2;      % ball mass      kg  
J = 0.5;    % beam inertia  kg m^2
```

```
r = X(1);  
q = X(2);  
dr = X(3);  
dq = X(4);  
g = 9.8;
```

```
M = [1.4*m, 0; 0, J + m*r*r];  
B1 = m*r*dq*dq - m*g*sin(q);  
B2 = T - 2*m*r*dr*dq - m*g*r*cos(q);
```

```
ddX = inv(M)*[B1; B2];  
dX = [dr; dq; ddX];
```

```
end
```

Change the main calling routine to include the feedback gains calculated here

```
% Ball & Beam System
% Spring 2024
% Homework #6

X = [0, 0, 0, 0]';
dt = 0.002;
t = 0;
n = 0;
y = [];

Kx = [-20.3209    25.8410   -2.3150    14.5000];
Kr = [-0.7209];

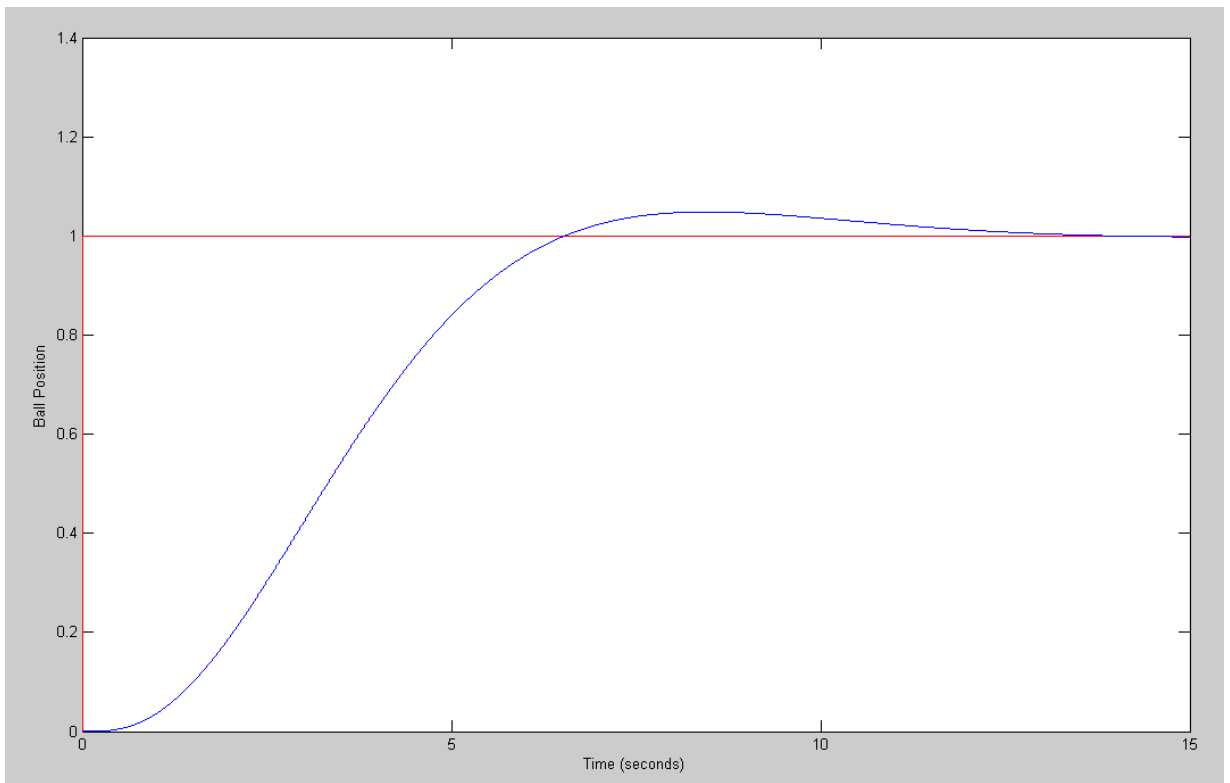
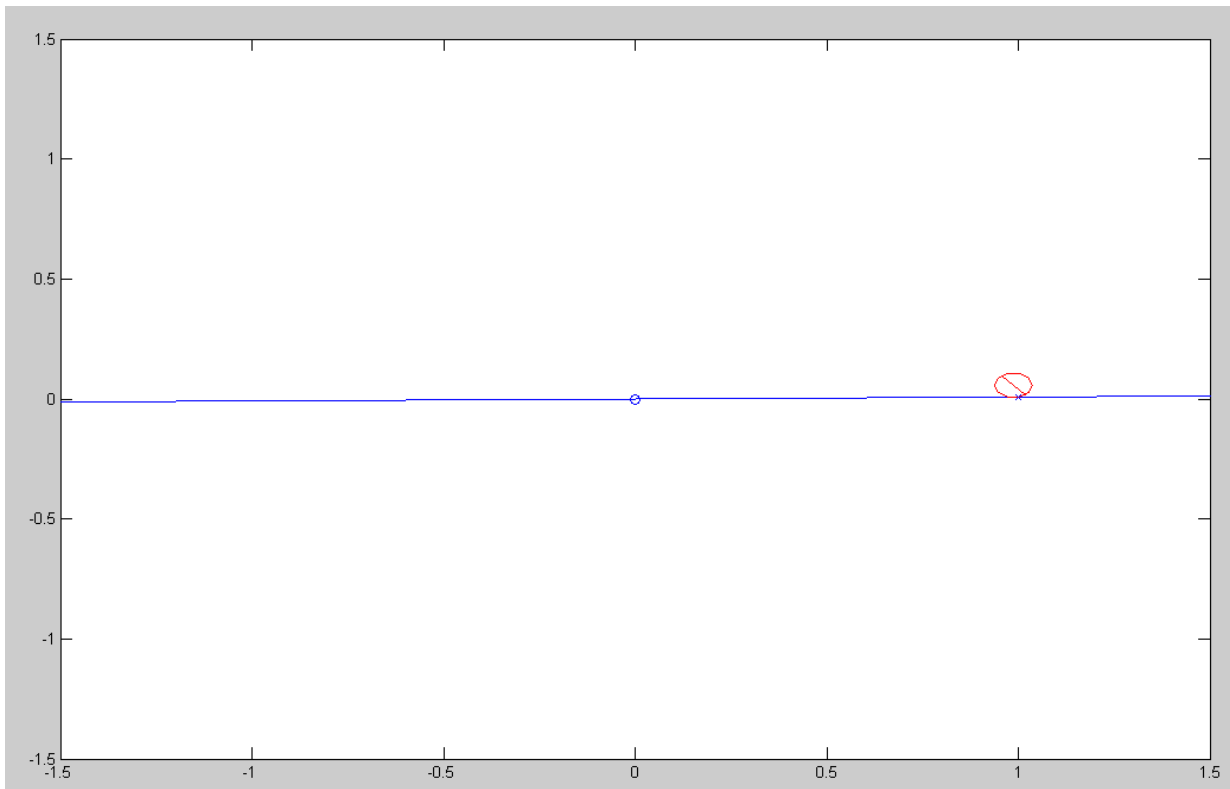
while(t < 15)
    Ref = 1;
    U = Kr*Ref - Kx*X;
    dX = BeamDynamics(X, U);

    X = X + dX * dt;
    t = t + dt;

    y = [y ; Ref, X(1)];
    n = mod(n+1,5);
    if(n == 0)
        BeamDisplay(X, Ref);
    end
end

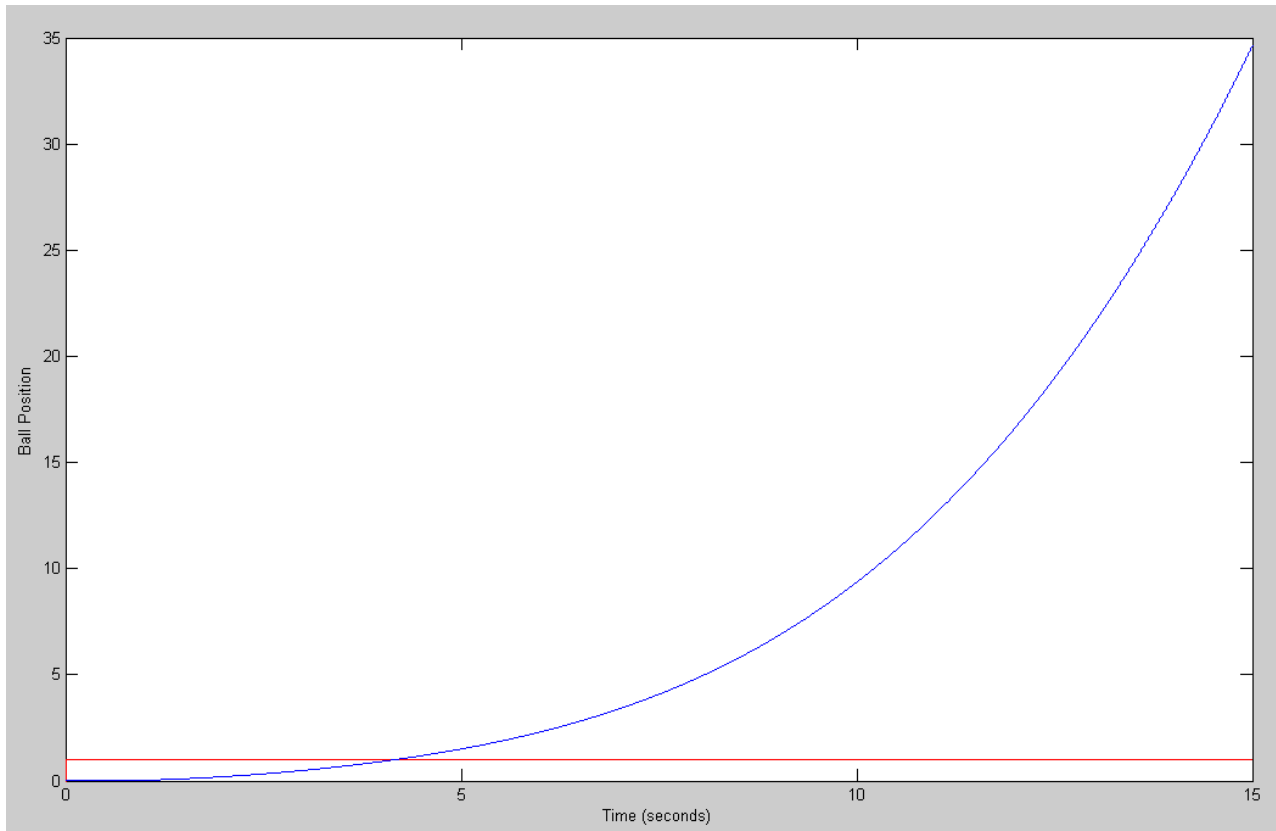
t = [1:length(y)]' * dt;

plot(t,y(:,1),'r',t,y(:,2),'b');
xlabel('Time (seconds)');
ylabel('Ball Position');
```



Servo Compensators

Problem 3) (10pt) Using your control law from problem #2, find the step response on the nonlinear simulation if the mass of the ball is changed from 2.0kg to 2.2kg.



Problem 4) (30pt) For the ball and beam system, design a feedback control law of the form

$$U = -K_Z Z - K_X X$$

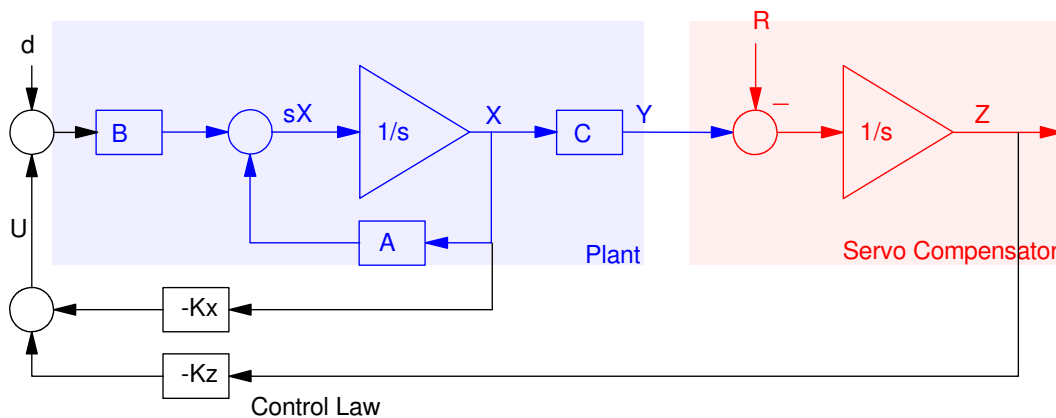
$$dZ = Y - \text{Ref}$$

where Y is the position of the ball. Pick the feedback gains so that the closed-loop system has

- A 2% settling time of 10 seconds,
- 5% overshoot for a step input, and
- A servo compensator designed to track a constant set point / reject a constant disturbance

Check the step response of the linear system in Matlab

Check the step response of the nonlinear system



Create the augmented system

$$\begin{bmatrix} sX \\ sZ \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} U + \begin{bmatrix} 0 \\ -1 \end{bmatrix} R$$

Find feedback gains for the augmented system

```
>> A5 = [A , zeros(4,1) ; C, 0]
```

```

      0      0      1.0000      0      0
      0      0      0      1.0000      0
      0     -7.0000      0      0      0
    -7.8400      0      0      0      0
     1.0000      0      0      0      0

```

```
>> B5u = [B;0]
```

```

      0
      0
      0
     0.4000
      0

```

```

>> B5r = [zeros(4,1) ; -1]

    0
    0
    0
    0
   -1

>> C5 = [1,0,0,0,0]

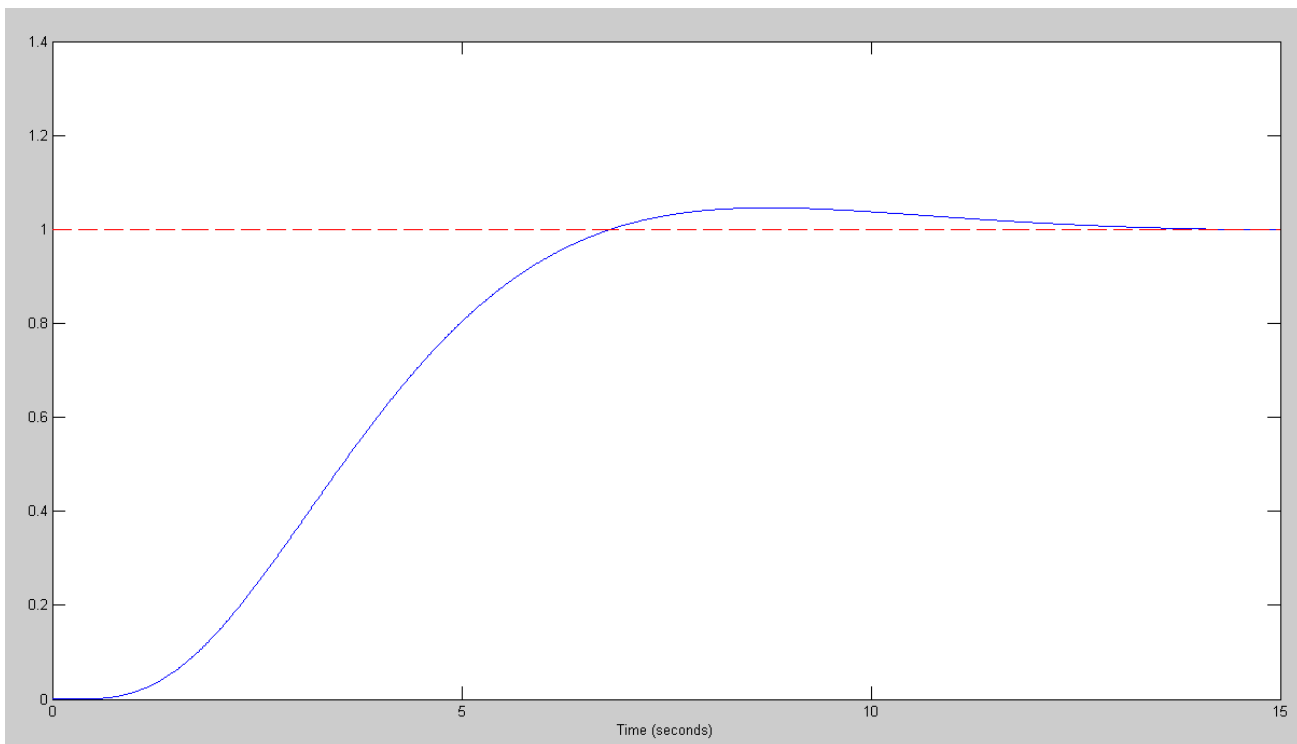
    1    0    0    0    0

>> K5 = ppl(A5, B5u, [-0.4+0.42j,-0.4-0.42j,-2,-3,-4])

K5 =  -29.5809   83.8410  -17.0813   24.5000  -2.8834

>> G = ss(A5 - B5u*K5, B5r, C5, 0);
>> t = [0:0.01:15]';
>> y = step(G,t);
>> Ref = 0*t + 1;
>> plot(t,y,'b',t,Ref,'r--')
>> xlabel('Time (seconds)');
>>

```



Modify the nonlienar simulation to include the servo compenesator

```
% Ball & Beam System
% Spring 2024
% Homework #6

X = [0, 0, 0, 0]';
dt = 0.002;
t = 0;
n = 0;
y = [];

Kx = [-29.5809    83.8410   -17.0813    24.5000];
Kz = [-2.8834];
Z = 0;

while(t < 15)
    Ref = 1*(sin(0.1*t) > 0);
    dZ = X(1) - Ref;

    U = -Kz*Z - Kx*X;
    dX = BeamDynamics(X, U);

    X = X + dX * dt;
    Z = Z + dZ * dt;

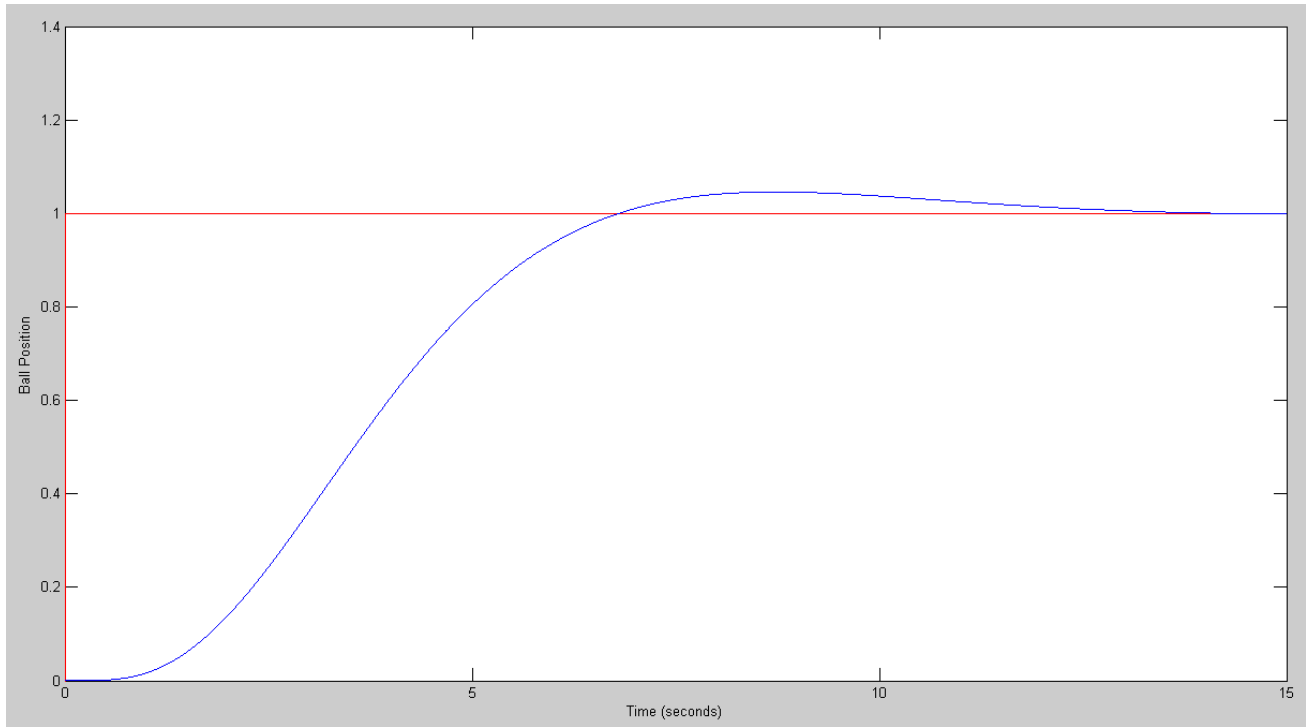
    t = t + dt;

    y = [y ; Ref, X(1)];
    n = mod(n+1,5);
    if(n == 0)
        BeamDisplay(X, Ref);
    end
end

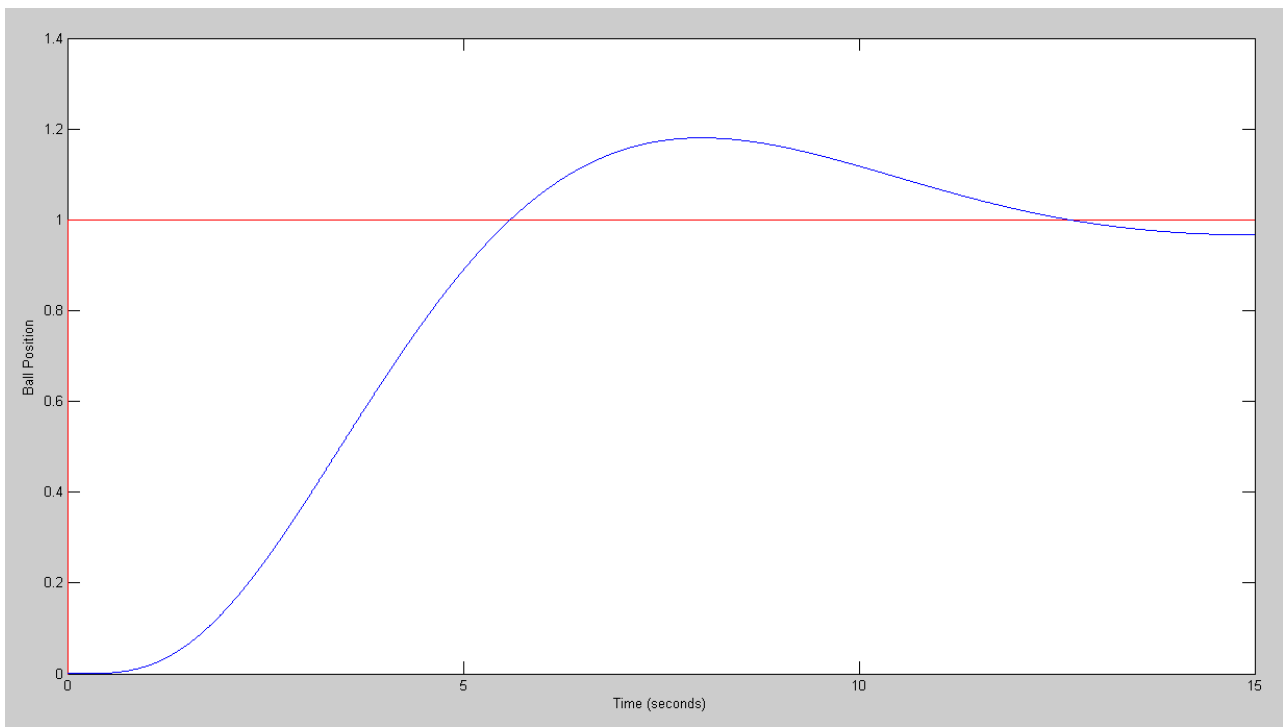
t = [1:length(y)]' * dt;

plot(t,y(:,1),'r',t,y(:,2),'b');
xlabel('Time (seconds)');
ylabel('Ball Position');
```

Take the step response when $m = 2.00\text{kg}$ (nominal)



Take the step response when $m = 2.2\text{kg}$



>

