

ECE 463/663 - Homework #3

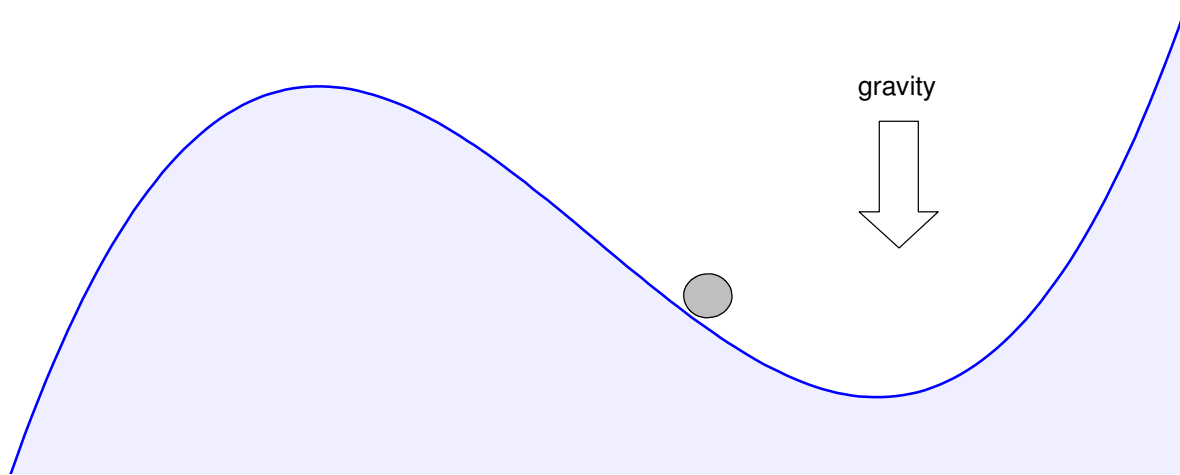
Block Diagrams and LaGrangian Dynamics. Due Monday, February 10th

LaGrangian Dynamics

A 1kg ball is rolling on a surface defined as

$$y = 0.2 \cdot (x - 2)(x)(x + 2)$$

Find the dynamics



1) Determine the kinetic and potential energy of this ball as a function of x : Gravity is in the $-y$ direction.

$$y = 0.2(x^3 - 4x)$$

$$\dot{y} = (0.6x^2 - 0.8)\dot{x}$$

Assuming a solid sphere:

Potential Energy

$$PE = mgh = 0.2g(x^3 - 4x)$$

kinetic energy (solid sphere)

$$KE = 0.7m(\dot{x}^2 + \dot{y}^2)$$

$$KE = 0.7\left(\dot{x}^2 + ((0.6x^2 - 0.8)\dot{x})^2\right)$$

$$KE = 0.7(1 + 0.36x^4 - 0.96x^2 + 0.64)\dot{x}^2$$

2) Determine the dynamics for this ball as it rolls on this surface

The LaGrangian is then

$$L = KE - PE$$

$$L = 0.7(1 + 0.36x^4 - 0.96x^2 + 0.64)\dot{x}^2 - 0.2g(x^3 - 4x)$$

$$L = (0.252x^4 - 0.672x^2 + 1.148)\dot{x}^2 - 0.2g(x^3 - 4x)$$

The force on the ball is

$$F = 0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \left(\frac{\partial L}{\partial x} \right)$$

$$0 = \frac{d}{dt} ((0.504x^4 - 1.344x^2 + 2.296)\dot{x})$$

$$-(1.008x^3 - 1.344x)\dot{x}^2 - 0.6gx^2 + 0.8g$$

$$0 = (0.504x^4 - 1.344x^2 + 2.296)\ddot{x}$$

$$+(2.0016x^3 - 2.688x)\dot{x}^2$$

$$-1.008x^3\dot{x}^2 + 1.344x\dot{x}^2 - 0.6gx^2 + 0.8g$$

$$0 = (0.504x^4 - 1.344x^2 + 2.296)\ddot{x} + 1.008x^3\dot{x}^2 - 1.344x\dot{x}^2 - 0.6gx^2 + 0.8g$$

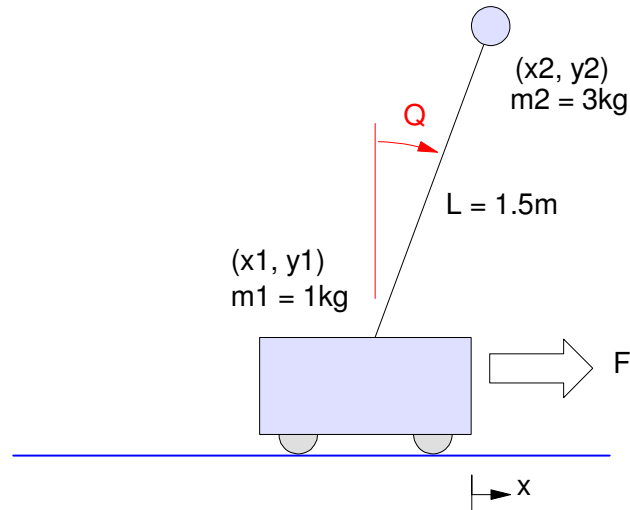
The acceleration on the ball is then

$$\ddot{x} = - \left(\frac{1.008x^3\dot{x}^2 - 1.344x\dot{x}^2 - 0.6gx^2 + 0.8g}{0.504x^4 - 1.344x^2 + 2.296} \right)$$

1) (30pt) Derive the dynamics for an inverted pendulum where

- $m_1 = 1\text{kg}$ (mass of cart)
- $m_2 = 3\text{kg}$ (mass of ball)
- $L = 1.5\text{m}$ (length of arm)

Fine the linearized dynamics at $x = 0, \theta = 0$



Mass #1 ($m_1 = 1\text{kg}$)

$$x_1 = x \quad y_1 = 0$$

$$\dot{x}_1 = \dot{x} \quad \dot{y}_1 = 0$$

$$PE = 0$$

$$KE = \frac{1}{2}mv^2 = 0.5\dot{x}^2$$

Mass #2 ($m_2 = 3\text{kg}$)

$$x_2 = x + 1.5\sin\theta \quad y_2 = 1.5\cos\theta$$

$$\dot{x}_2 = \dot{x} + 1.5\cos\theta\dot{\theta} \quad \dot{y}_2 = -1.5\sin\theta\dot{\theta}$$

$$PE = m_2gy_2 = 3 \cdot g \cdot 1.5\cos\theta$$

$$KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$KE = \frac{3}{2}\left(\left(\dot{x} + 1.5\cos\theta\dot{\theta}\right)^2 + \left(-1.5\sin\theta\dot{\theta}\right)^2\right)$$

$$KE = 1.5\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta$$

The LaGrangian is then

$$L = KE - PE$$

$$L = (0.5\dot{x}^2) + \left(1.5\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta\right) - (4.5g\cos\theta)$$

To find the dynamics, use the Euler LaGrange equation

$$L = \left(2\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta\right) - (4.5g\cos\theta)$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right)$$

$$F = \frac{d}{dt}\left(4\dot{x} + 4.5\cos\theta\dot{\theta}\right) - (0)$$

$$F = 4\ddot{x} + 4.5\cos\theta\ddot{\theta} - 4.5\sin\theta\dot{\theta}^2$$

$$L = \left(2\dot{x}^2 + 3.375\dot{\theta}^2 + 4.5\dot{x}\dot{\theta}\cos\theta\right) - (4.5g\cos\theta)$$

$$T = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right)$$

$$T = \frac{d}{dt}\left(6.75\dot{\theta} + 4.5\cos\theta\dot{x}\right) - \left(-4.5\sin\theta\dot{x}\dot{\theta} + 4.5g\sin\theta\right)$$

$$T = 6.75\ddot{\theta} + 4.5\cos\theta\ddot{x} - 4.5\sin\theta\dot{x}\dot{\theta} + 4.5\sin\theta\dot{x}\dot{\theta} - 4.5g\sin\theta$$

$$T = 6.75\ddot{\theta} + 4.5\cos\theta\ddot{x} - 4.5g\sin\theta$$

So, the dynamics are

$$F = 4\ddot{x} + 4.5\cos\theta\ddot{\theta} - 4.5\sin\theta\dot{\theta}^2$$

$$T = 6.75\ddot{\theta} + 4.5\cos\theta\ddot{x} - 4.5g\sin\theta$$

In Matrix form

$$\begin{bmatrix} 4 & 4.5\cos\theta \\ 4.5\cos\theta & 6.75 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F \\ T \end{bmatrix} + \begin{bmatrix} 4.5\sin\theta\dot{\theta}^2 \\ 4.5g\sin\theta \end{bmatrix}$$

Linearizing about zero with $T = 0$

$$\begin{bmatrix} 4 & 4.5 \\ 4.5 & 6.75 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 4.5g\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.667 \end{bmatrix} F + \begin{bmatrix} -3g\theta \\ 2.667g\theta \end{bmatrix}$$

Putting this in state-space form

$$s \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -29.4 & 0 & 0 \\ 0 & 26.133 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -0.667 \end{bmatrix} F$$

The poles are at

```
>> A = [0,0,1,0;0,0,0,1;0,-29.4,0,0;0,26.133,0,0]
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```
      0      0      1.0000      0
      0      0      0      1.0000
      0 -29.4000      0      0
      0  26.1330      0      0
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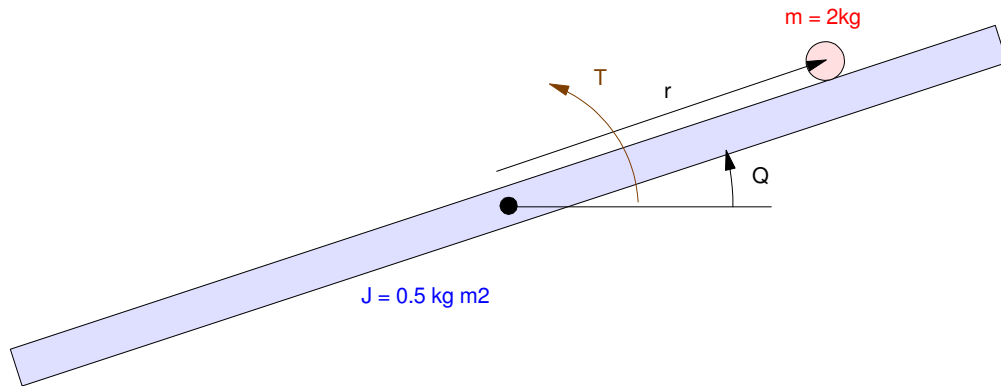
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>> eig(A)
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      0
      0
      5.1120
     -5.1120
```

2) (30pt) Derive the dynamics for a ball and beam system where

- $J = 0.5 \text{ kg m}^2$ (the inertia of the beam)
- $m = 2 \text{ kg}$ (the mass of the ball)

Find the linearized dynamics at $r = 1.0 \text{ m}$, $\theta = 0$



Position of the ball:

$$x_1 = r \cos \theta$$

$$y_1 = r \sin \theta$$

$$\dot{x}_1 = \dot{r} \cos \theta - r \sin \theta \dot{\theta}$$

$$\dot{y}_1 = \dot{r} \sin \theta + r \cos \theta \dot{\theta}$$

The potential and kinetic energy. Assuming a solid sphere with radius 5mm (0.005m)

$$J = \frac{2}{5} m r^2 \quad r = \text{radius here}$$

$$x = r \theta \quad x = \text{displacement here}$$

$$\dot{x} = r \dot{\theta}$$

$$KE = \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} \left(\frac{2}{5} m r^2 \right) \left(\frac{\dot{x}}{r} \right)^2 = \frac{1}{5} m \dot{x}^2 \quad \text{rotational KE for a solid sphere rolling}$$

This gives (using r for displacement along the beam for x)

$$PE = mgy_1 = mgr \sin \theta = 2gr \sin \theta$$

$$KE = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{5} m \dot{r}^2$$

$$KE = 0.25 \dot{\theta}^2 + (\dot{x}_1^2 + \dot{y}_1^2) + 0.4 \dot{r}^2$$

$$KE = 0.25 \dot{\theta}^2 + \left((\dot{r} \cos \theta - r \sin \theta \dot{\theta})^2 + (\dot{r} \sin \theta + r \cos \theta \dot{\theta})^2 \right) + 0.4 \dot{r}^2$$

$$KE = 0.25 \dot{\theta}^2 + (\dot{r}^2 + r^2 \dot{\theta}^2) + 0.4 \dot{r}^2$$

$$KE = (0.25 + r^2) \dot{\theta}^2 + 1.4 \dot{r}^2$$

The LaGrangian is then

$$L = KE - PE$$

$$L = \left((0.25 + r^2)\dot{\theta}^2 + 1.4\dot{r}^2 \right) - (2gr \sin \theta)$$

$$L = \left(0.25\dot{\theta}^2 + r^2\dot{\theta}^2 + 1.4\dot{r}^2 \right) - (2gr \sin \theta)$$

Force on the Ball

$$L = \left(0.25\dot{\theta}^2 + r^2\dot{\theta}^2 + 1.4\dot{r}^2 \right) - (2gr \sin \theta)$$

$$F = \frac{d}{dt} \left(\frac{\partial L}{\partial r'} \right) - \left(\frac{\partial L}{\partial r} \right)$$

$$F = \frac{d}{dt} (2.8\dot{r}) - (2r\dot{\theta}^2 - 2g \sin \theta)$$

$$F = 2.8\ddot{r} - 2r\dot{\theta}^2 + 2g \sin \theta$$

Torque on the Beam

$$L = \left(0.25\dot{\theta}^2 + r^2\dot{\theta}^2 + 1.4\dot{r}^2 \right) - (2gr \sin \theta)$$

$$T = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \left(\frac{\partial L}{\partial \theta} \right)$$

$$T = \frac{d}{dt} (0.5\dot{\theta} + 2r^2\dot{\theta}) - (-2gr \cos \theta)$$

$$T = 0.5\ddot{\theta} + 2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} + 2gr \cos \theta$$

Putting it together

$$\begin{bmatrix} 2.8 & 0 \\ 0 & 0.5 + 2r^2 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2r\dot{\theta}^2 - 2g \sin \theta \\ -4r\dot{r}\dot{\theta} - 2gr \cos \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

Linearizing at $r = 1.0\text{m}$

$$\begin{bmatrix} 2.8 & 0 \\ 0 & 2.5 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -2g\theta \\ -2gr \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T$$

In State-Space form

$$s \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -7.84 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \end{bmatrix} T$$

The open-loop system is unstable

```
>> A = [0,0,1,0;0,0,0,1;0,-7,0,0;-7.84,0,0,0]
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```
A =
```

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      0      0      1.0000      0
      0      0      0      1.0000
      0     -7.0000      0      0
    -7.8400      0      0      0
```

```
>> eig(A)
```

```
ans =
```

```

-2.7218
-0.0000 + 2.7218i
-0.0000 - 2.7218i
 2.7218
```