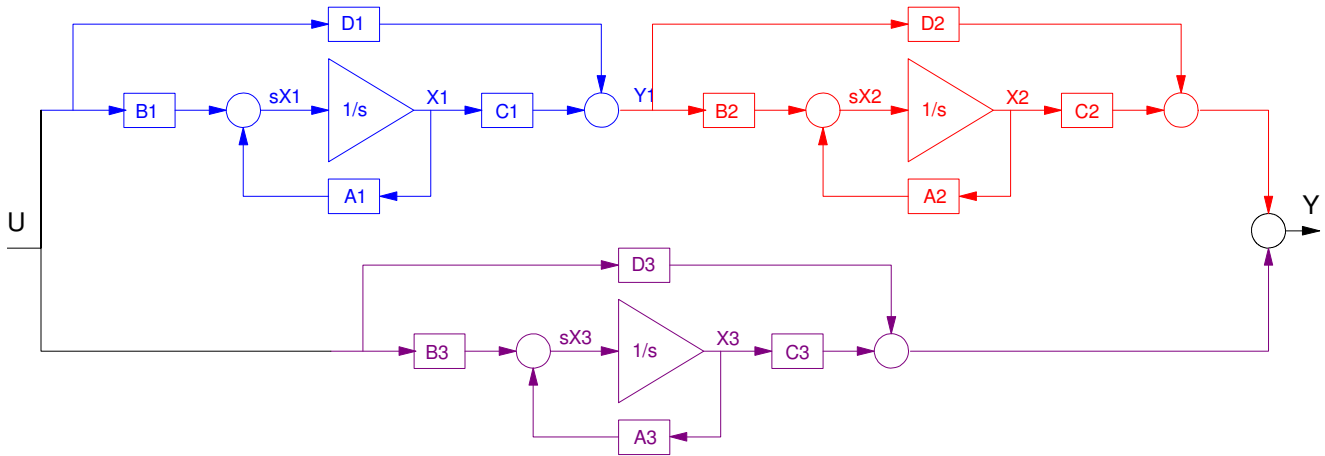


ECE 463/663 - Homework #2

State-Space, Eigenvalues, Canonical Forms

State-Space

1) Give the state-space model for the following block diagram



The dynamics are

$$sX_1 = B_1 U + A_1 X_1$$

$$sX_2 = B_2(C_1 X_1 + D_1 U) + A_2 X_2$$

$$sX_3 = B_3 U + A_3 X_3$$

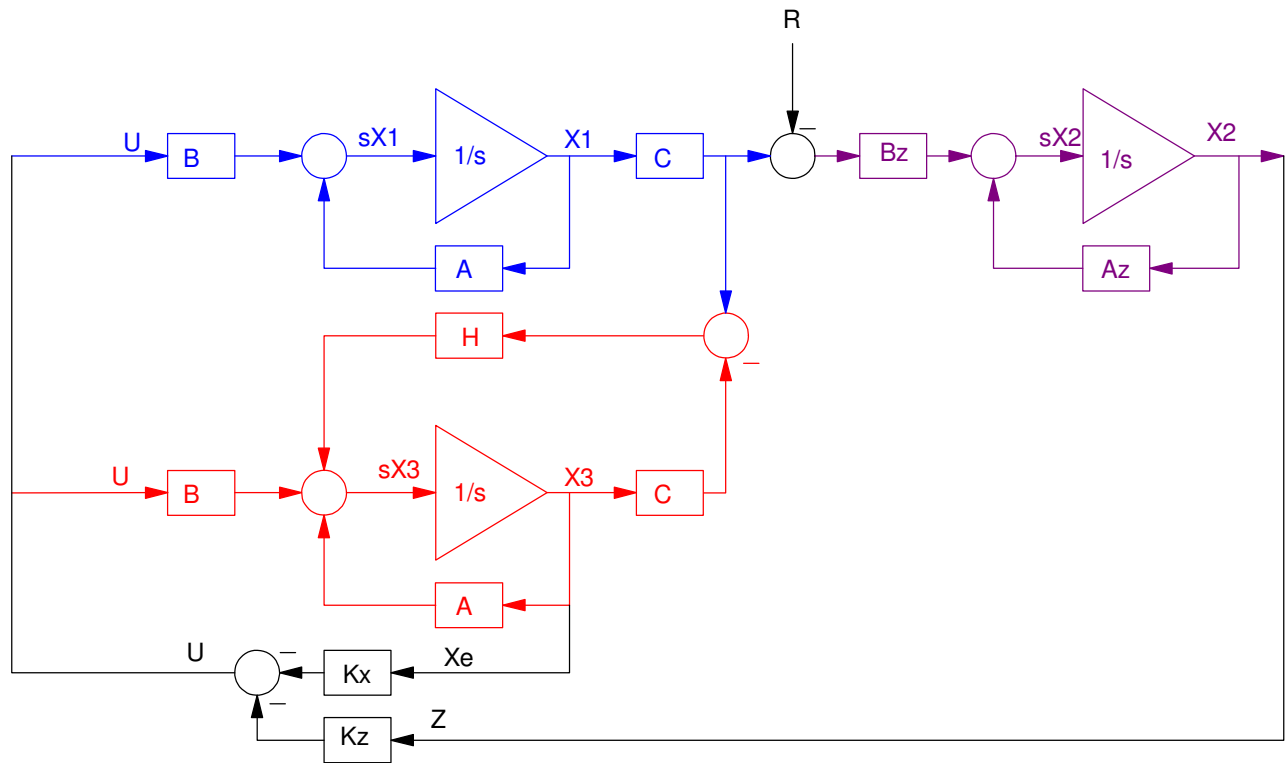
$$Y = C_2 X_2 + D_2(C_1 X_1 + D_1 U) + C_3 X_3 + D_3 U$$

Placing in matrix form

$$s \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ B_2 C_1 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \\ B_3 \end{bmatrix} U$$

$$Y = \begin{bmatrix} D_2 C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + [D_2 D_1 + D_3] U$$

2) Give the state-space model for the following block diagram



Write the dynamics

$$sX_1 = AX_1 + B(-K_xX_3 - K_zX_2)$$

$$sX_2 = A_zX_2 + B_z(CX_1 - R)$$

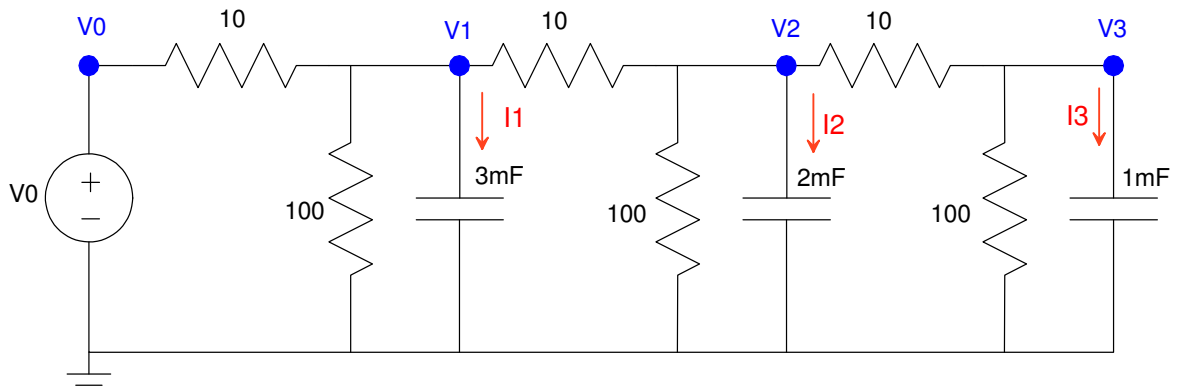
$$sX_3 = AX_3 + B(-K_xX_3 - K_zX_2) + H(CX_1 - CX_3)$$

Place in matrix form

$$s \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} A & -BK_z & -BK_x \\ B_zC & A_z & 0 \\ HC & -BK_z & A - BK_x - HC \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -B_z \\ 0 \end{bmatrix} R$$

State-Space and Eigenvalues

Problems 3-5: Use the following RC circuit



3) Write the differential equations that describe this circuit

$$I_1 = 0.003sV_1 = \left(\frac{V_0 - V_1}{10} \right) + \left(\frac{V_2 - V_1}{10} \right) - \left(\frac{V_1}{100} \right)$$

$$I_2 = 0.002sV_2 = \left(\frac{V_1 - V_2}{10} \right) + \left(\frac{V_3 - V_2}{10} \right) - \left(\frac{V_2}{100} \right)$$

$$I_3 = 0.001sV_3 = \left(\frac{V_2 - V_3}{10} \right) - \left(\frac{V_3}{100} \right)$$

4) Find the transfer function from V_0 to V_3

Group terms and simplify

$$sV_1 = 33.33V_0 - 70V_1 + 33.33V_2$$

$$sV_2 = 50V_1 - 105V_2 + 50V_3$$

$$sV_3 = 100V_2 - 110V_3$$

Place in matrix form

$$s \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -70 & 33.33 & 0 \\ 50 & -105 & 50 \\ 0 & 100 & -110 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 33.33 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Place into Matlab and find the transfer function

```
>> A = [-70, 33.33, 0; 50, -105, 50; 0, 100, -110]
```

```

-70.0000    33.3300         0
 50.0000   -105.0000    50.0000
   0    100.0000   -110.0000

```

```
>> B = [33.33; 0; 0]
```

```

33.3300
   0
   0

```

```
>> C = [0, 0, 1]
```

```

0         0         1

```

```
>> G = ss(A, B, C, 0);
```

```
>> zpk(G)
```

```

166650

```

```

-----
(s+18.27) (s+81.15) (s+185.6)

```

```
>>
```

5) Assume $V_0=0$ and $V_3(0) = 1$. What initial conditions on V_1 and V_2 result in

- $V_3(t)$ decaying as fast as possible?
- $V_3(t)$ decaying as slow as possible?

This is an eigenvalue / eigenvector problem

```
>> [M,V] = eig(A)
```

Eigenvectors

-0.3993	0.6380	0.1713
-0.6197	-0.2134	-0.5940
-0.6756	-0.7398	0.7860

Eigenvalues

-18.2731	0	0
0	-81.1501	0
0	0	-185.5768

The slow mode corresponds to the first eigenvector. Scaling it so $V_3(0) = 1$:

```
>> X0 = M(:,1) / M(3,1)
```

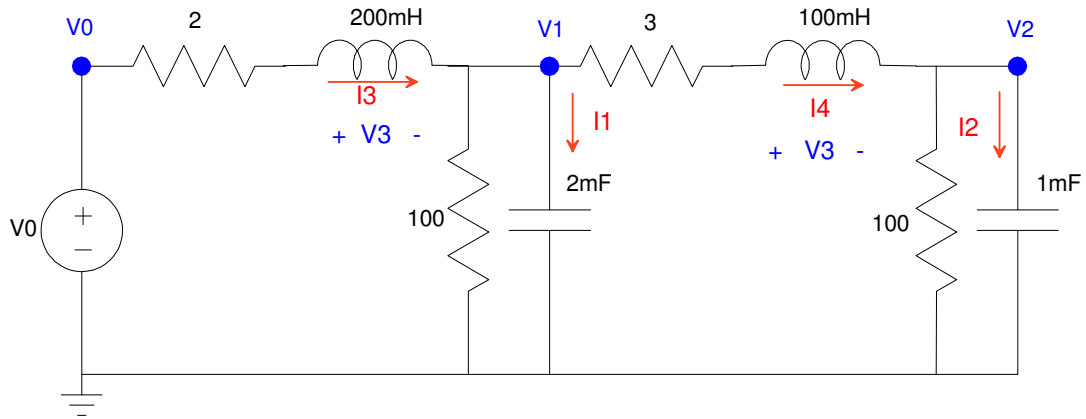
V1	0.5910
V2	0.9173
V3	1.0000

The fast mode corresponds to the third eigenvector. Scaling it so $V_3(0) = 1$

```
>> X0 = M(:,3) / M(3,3)
```

V1	0.2179
V2	-0.7558
V3	1.0000

Problems 6-8: Use the following RLC circuit



6) Write the differential equations that describe this circuit

$$I_1 = 0.002sV_1 = I_3 - \left(\frac{V_1}{100}\right) - I_4$$

$$I_2 = 0.001sV_2 = I_4 - \left(\frac{V_2}{100}\right)$$

$$V_3 = 0.2sI_3 = V_0 - 2I_3 - V_1$$

$$V_4 = 0.1sI_4 = V_1 - 3I_4 - V_2$$

7) Find the transfer function from V_0 to V_2

Group terms and simplify

$$sV_1 = -5V_1 + 500I_3 - 500I_4$$

$$sV_2 = -10V_2 + 1000I_4$$

$$sI_3 = 5V_0 - 10I_3 - 5V_1$$

$$sI_4 = 10V_1 - 30I_4 - 10V_2$$

Place into matrix form

$$s \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 500 & -500 \\ 0 & -10 & 0 & 1000 \\ -5 & 0 & -10 & 0 \\ 10 & -10 & 0 & -30 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix} V_0$$

$$Y = V_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \end{bmatrix}$$

Put into Matlab and find the transfer function

```
>> A = [-5,0,500,-500;0,-10,0,1000;-5,0,-10,0;10,-10,0,-30]
```

```

-5      0      500     -500
 0     -10      0     1000
-5      0     -10      0
10     -10      0     -30
```

```
>> B = [0;0;5;0]
```

```

0
0
5
0
```

```
>> C = [0,1,0,0];
```

```
>> D = 0;
```

```
>> G = ss(A,B,C,D);
```

```
>> zpk(G)
```

25000000

(s^2 + 18.34s + 1661) (s^2 + 36.66s + 1.612e004)

8) Assume $V_0=0$ and $V_2(0) = 1$. What initial conditions result in

- $V_2(t)$ decaying as fast as possible?
- $V_2(t)$ decaying as slow as possible?

This is an eigenvalue / eigenvector problem. Finding them using Matlab

```
>> [M,V] = eig(A)

eigenvectors

    0.5034 - 0.0360i    0.5034 + 0.0360i    0.6412 + 0.0653i    0.6412 - 0.0653i
   -0.8563           -0.8563           0.7597           0.7597
    0.0027 + 0.0199i    0.0027 - 0.0199i   -0.0099 + 0.0805i   -0.0099 - 0.0805i
    0.0071 - 0.1076i    0.0071 + 0.1076i    0.0006 + 0.0302i    0.0006 - 0.0302i

eigenvalues

1.0e+002 *
   -0.1833 + 1.2562i         0              0              0
         0          -0.1833 - 1.2562i         0              0
         0              0          -0.0917 + 0.3971i         0
         0              0              0          -0.0917 - 0.3971i
```

The slow mode is the 3rd & 4th column (real = cosine, imag = -sine). Scaling so that $V_2(0) = 1$

```
>> X0 = real(M(:,3)) / M(2,3)

V1    0.8441
V2    1.0000
I3   -0.0130
I4    0.0008
```

The fast mode is the 1st & second column. Scaling so that $V_2(0) = 1$

```
>> X0 = real(M(:,1)) / M(2,1)

V1   -0.5878
V2    1.0000
V3   -0.0032
V4   -0.0083
```

Canonical Forms

9) Let the transfer function from X to Y be

$$Y = \left(\frac{10(s+2)}{s(s+5)(s+20)} \right) X$$

Specify the {A, B, C, D} matrix that describe this system in state-space form:

In controller canonical form

```
>> poly([0,-5,-20])  
ans =      1      25     100      0  
  
>> A = [0,1,0;0,0,1;0,-100,-25]  
  
      0      1      0  
      0      0      1  
      0     -100     -25  
  
>> B = [0;0;1]  
  
      0  
      0  
      1  
  
>> C = [20,10,0]  
  
C =      20      10      0  
  
>> G = ss(A,B,C,D);  
>> zpk(G)  
  
      10 (s+2)  
-----  
s (s+5) (s+20)
```

In Jordan (diagonal) canonical form. Do a partial fraction expansion

$$\left(\frac{10(s+2)}{s(s+5)(s+20)} \right) = \left(\left(\frac{0.2}{s} \right) + \left(\frac{0.4}{s+5} \right) + \left(\frac{-0.6}{s+20} \right) \right)$$

$$sX = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -20 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0.2 & 0.4 & -0.6 \end{bmatrix} X$$

Similarity Transforms

10) Let X and Y be related by the following state-space model

$$sX = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} X$$

What is the state-space model if you do a change of variable:

$$Z = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} X$$

```
>> A = [-2,1,0;1,-2,1;0,1,-1]
      -2    1    0
      1   -2    1
      0    1   -1
```

```
>> B = [1;0;0]
      1
      0
      0
```

```
>> C = [1,2,3]
C =      1      2      3
```

```
>> Ti = [1,1,0;1,0,-1;1,1,1]
      1    1    0
      1    0   -1
      1    1    1
```

```
>> T = inv(Ti);
>> Az = inv(T)*A*T
      -2    0    1
      1   -2   -1
      1   -1   -1
```

```
>> Bz = inv(T)*B
      1
      1
      1
```

```
>> Cz = C*T
      0   -1    2
```

$$sZ = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix} Z + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix} Z$$