Root Locus in the z-Domain

ECE 461/661 Controls Systems

Jake Glower - Lecture #31

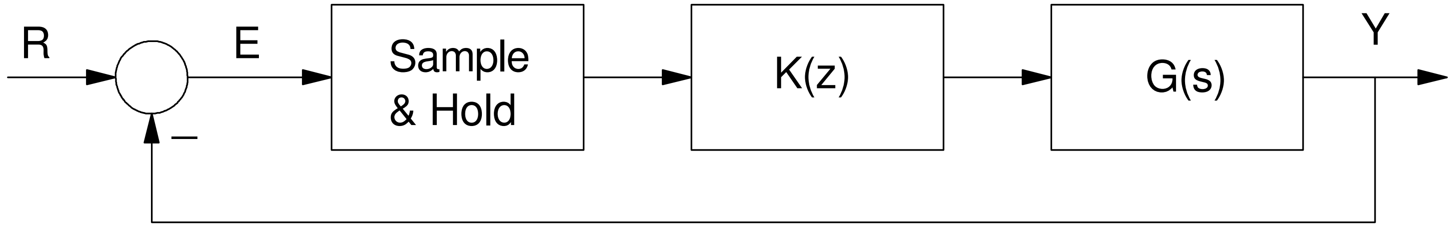
Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Root Locus in the z-Domain

Goal: Find K(z) for a "good" response

Note: Relative to the microcontroller

- The feedback system looks like it's a discrete-time system.
- It looks like it's in the z-domain.



Mathematically, the open-loop transfer function is

$$H(s)\,G(s)\,K(z)$$

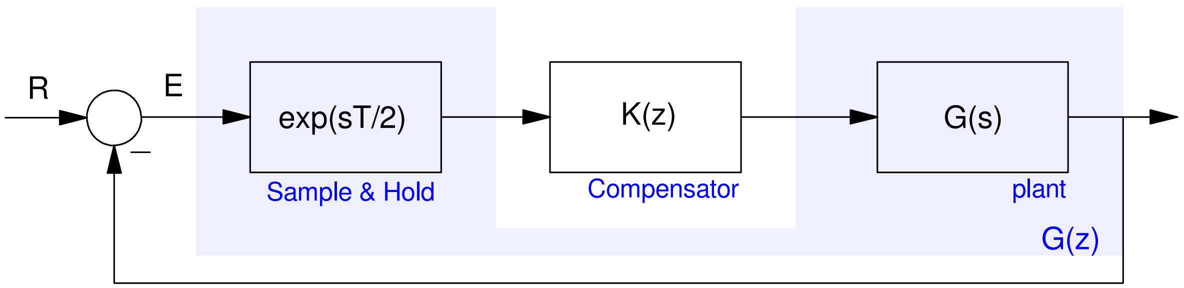
- H(s) Sample and Hold
- G(s) Analog Plant
- K(z) Digital Controller

H(s) looks like a 1/2 sample delay

$$H(s) \approx \exp\left(-\frac{sT}{2}\right)$$

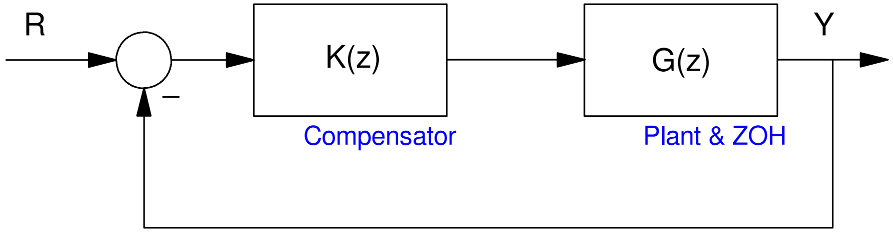
Lump H(s) and G(s) together

$$H(s)\,G(s) \Rightarrow G(z)$$



The closed-loop system is

$$Y = \left(\frac{GK}{1+GK}\right) R$$



If GK has zeros and poles:

$$GK = k\frac{z}{p}$$

the closed-loop transfer function becomes

$$Y = \left(\frac{kz}{p+kz}\right) E$$

The roots of the closed-loop system are:

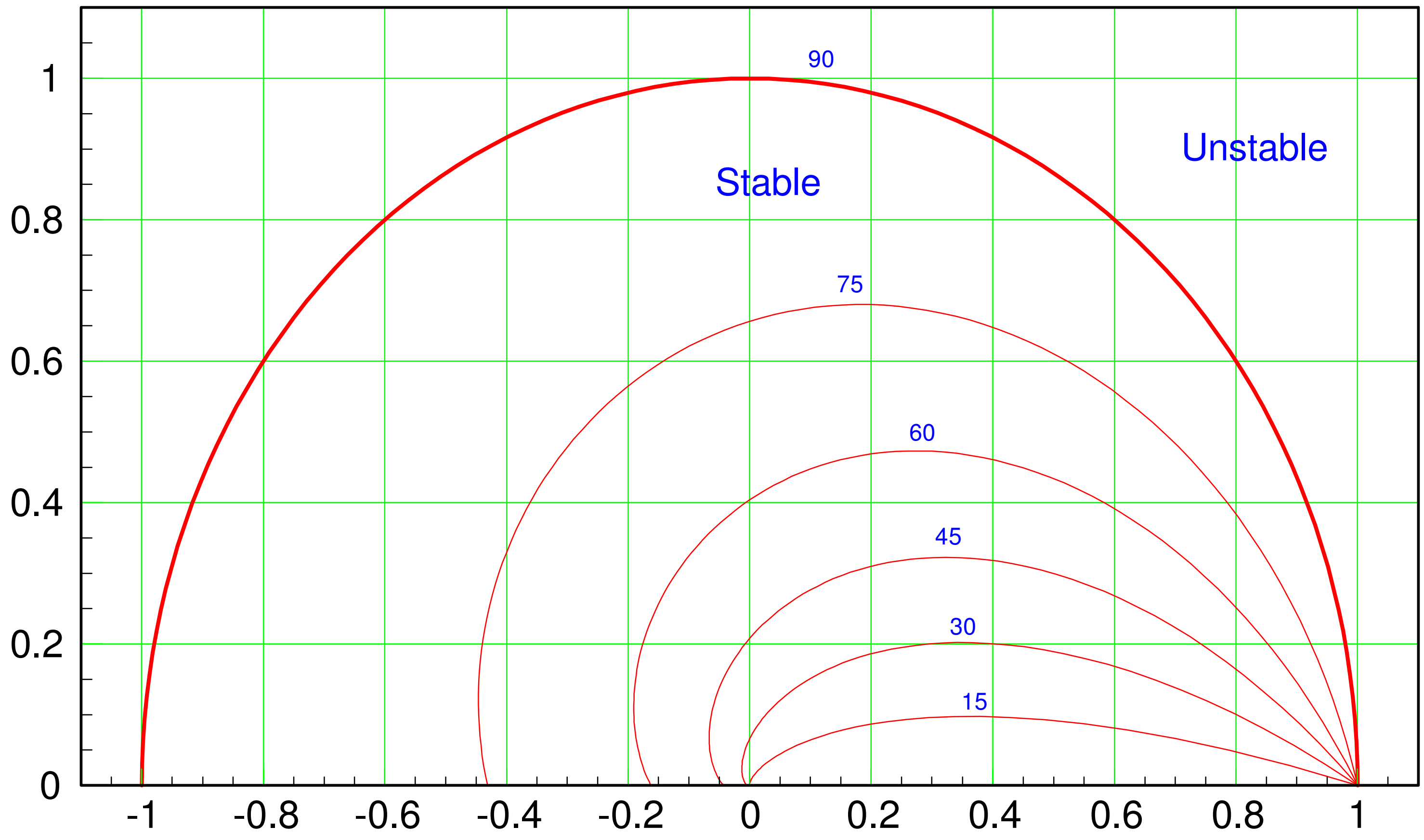
$$p(z) + k\,z(z) = 0$$

Note: This is identical to the s-plane

$$p(s) + k\,z(s) = 0$$

Root Locus in the z-plane

- Exactly the same as the s-plane
- The only difference is how to interprit the results: $z = e^{sT}$



2% Settling Time:

- Determined by the magnitude of the poles

$$z^k = 0.02$$

$$k = \frac{\ln(0.02)}{\ln(z)}$$

A pole at z = 0.95 has a 2% settling time of 73 samples (round up)

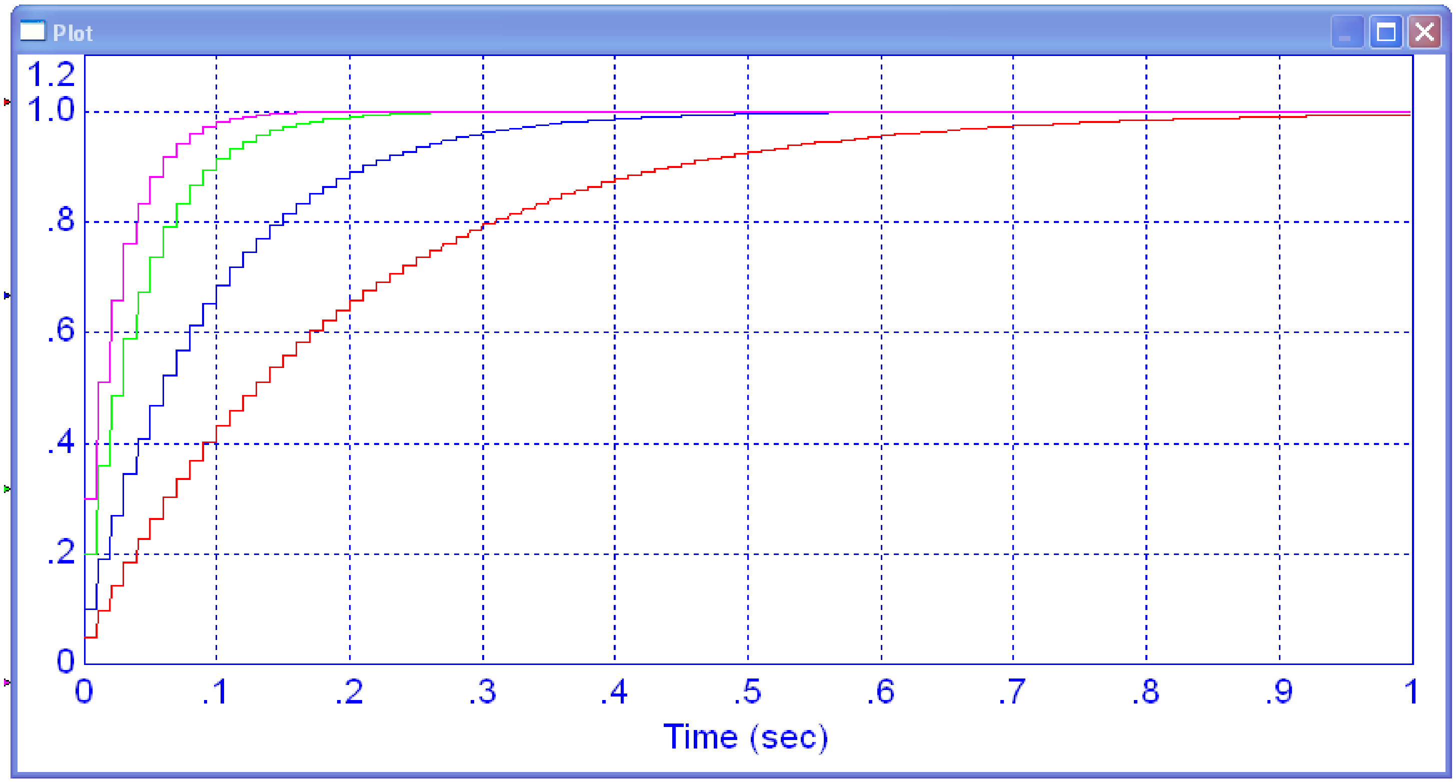
$$k = \frac{\ln(0.02)}{\ln(0.95)} = 72.26 \text{ samples}$$

A pole at z = 0.8 has a 2% settling time of 18 samples

$$k = \frac{\ln(0.02)}{\ln(0.8)} = 17.53 \text{ samples}$$

Similarly, for any pole, the amplitude tells you the settling time as

$$|z|^k = 0.02 \qquad k = \frac{\ln(0.02)}{\ln(|z|)}$$



Step Response: Pole at { 0.95 (red), 0.9 (blue), 0.8 (green), and 0.7 (pink). T = 0.01 second

Frequency of Oscillation:

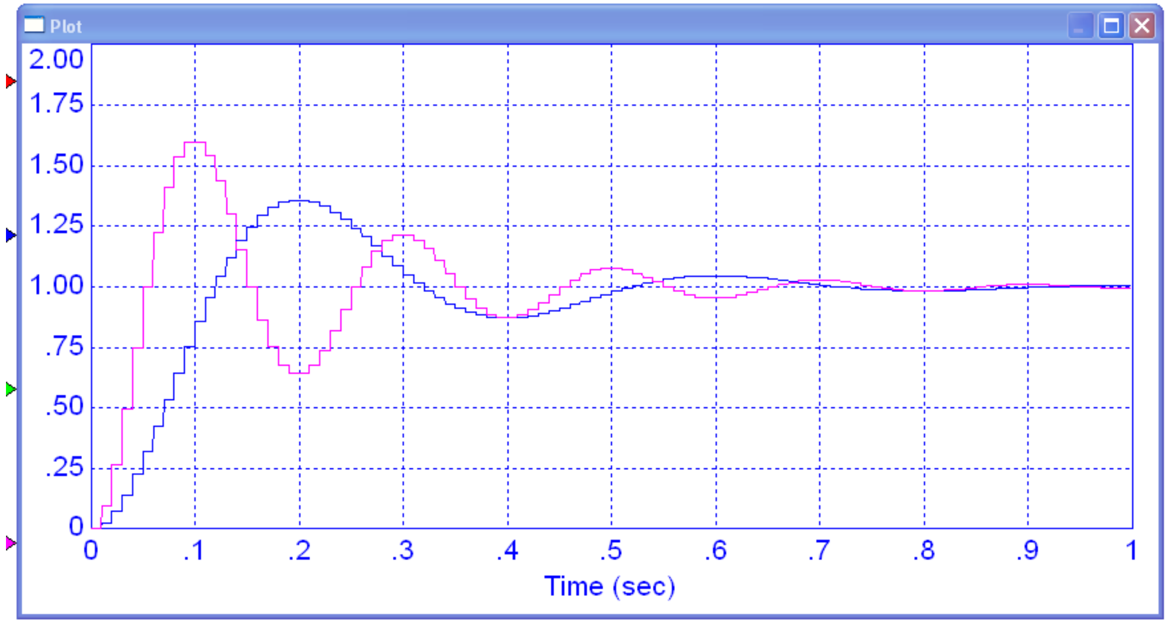
- Determined by the angle of the poles

$$\text{period} = \frac{360^0}{angle} \quad \text{samples}$$

Poles at

$0.95\angle \pm 9^0$ Period = 40 samples

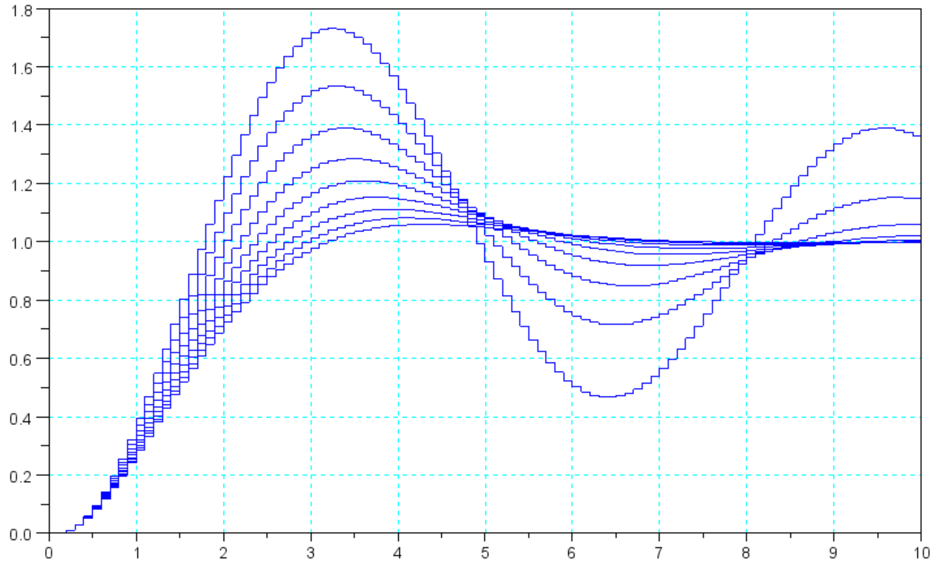
$0.95\angle \pm 18^0$ Period = 20 samples



Overshoot (Damping Ratio)

- Follows a log spiral
- Convert to the s-plane to find the damping ratio

Damping Ratio	s-plane	z-plane
0	$-1\angle\pm 90^0$	0.9950 + j0.0998
0.2	$-1\angle\pm 78.46^0$	0.9755 + j0.0959
0.4	$-1\angle\pm 66.42^0$	0.9568 + j0.0879
0.6	$-1\angle\pm 53.13^0$	0.9388 + j0.0753
0.8	$-1\angle\pm 36.86^0$	0.9214 + j0.0553
1	$-1\angle\pm 0^0$	0.9048 + j0



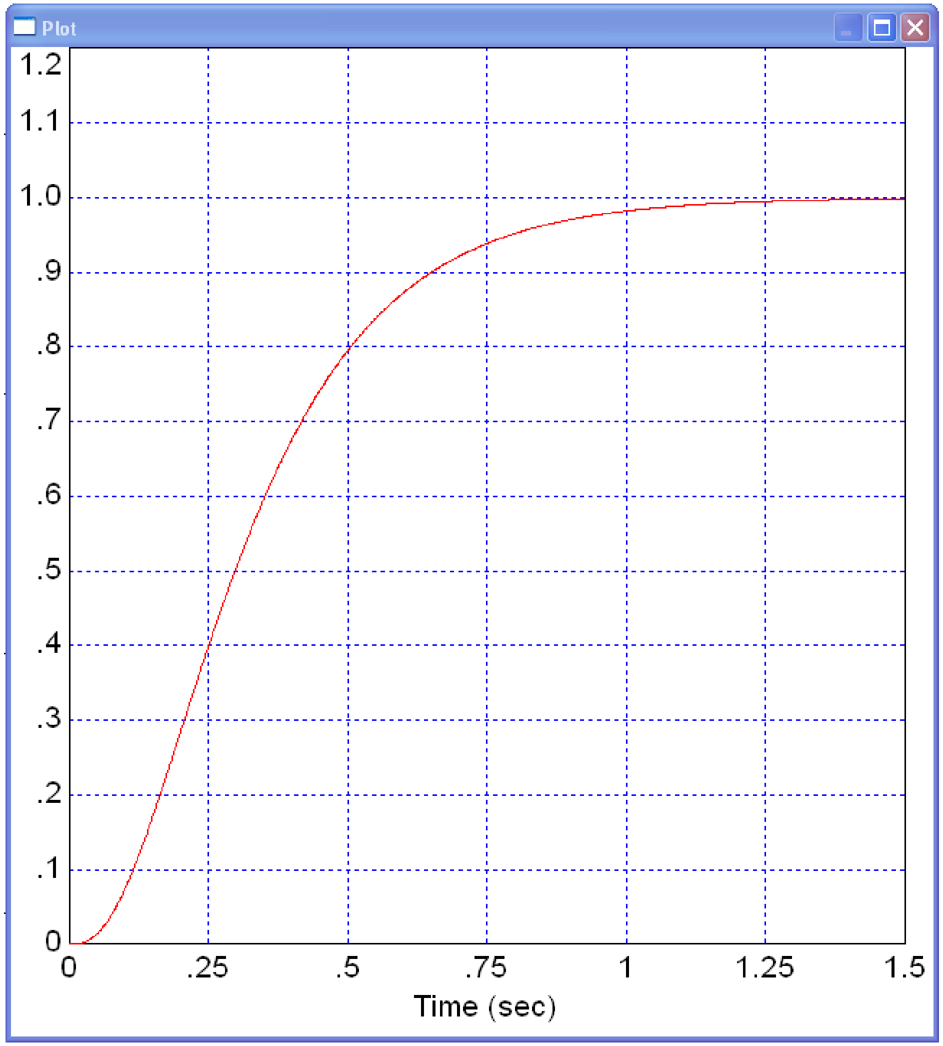
Gain Compensation in the z-Plane

Assume

$$G(s) = \left(\frac{1000}{(s+5)(s+10)(s+20)}\right)$$

Find K(z) = k

- T = 10ms
- No overshoot,
- 20% overshoot, and
- The maximum gain for stability.

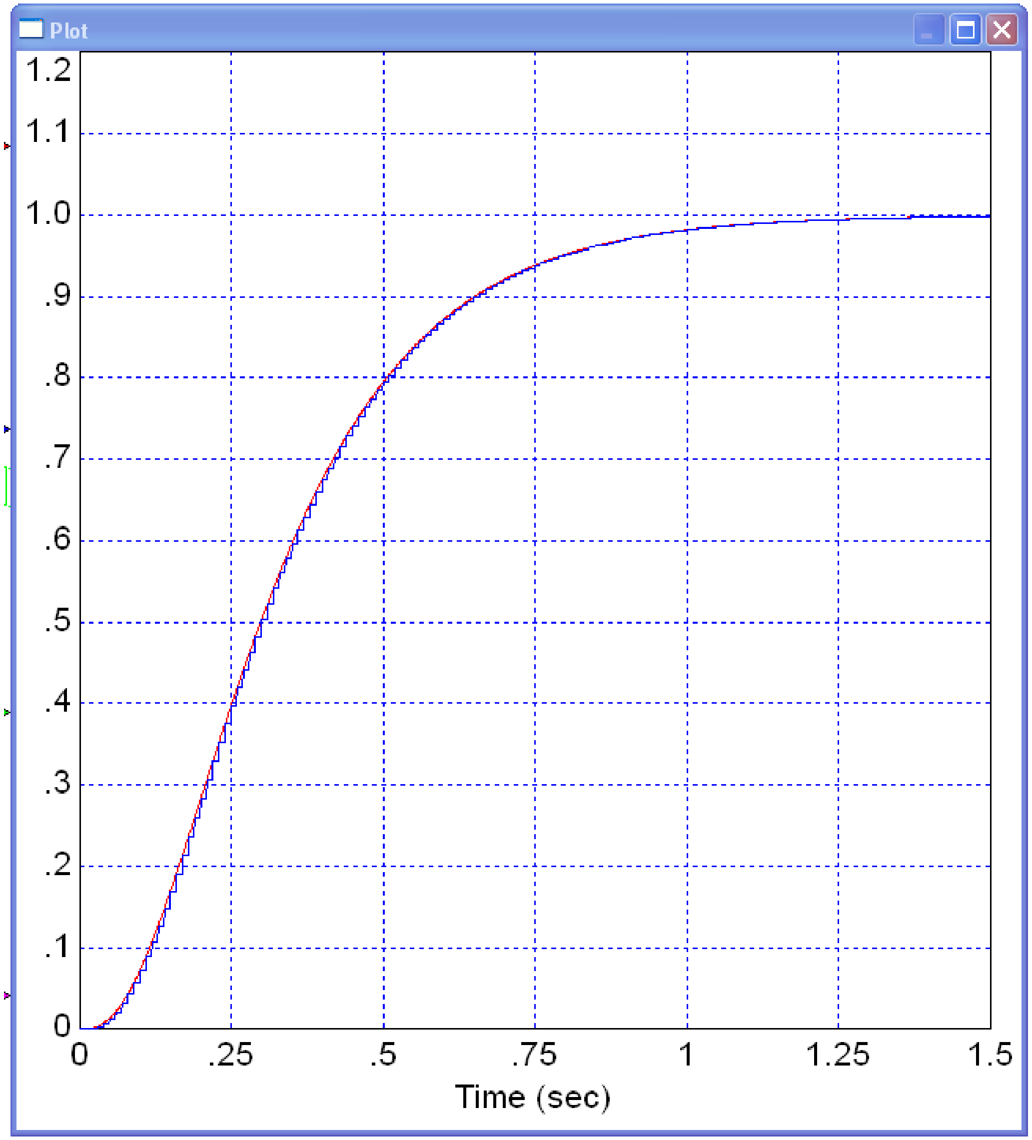


Step 1: Convert G(s) to G(z).

$$G(s) = \left(\frac{1000}{(s+5)(s+10)(s+20)} \right)$$

With T = 0.01 second

$$G(z) \approx \left(\frac{0.0008413z}{(z-0.9512)(z-0.9048)(z-0.8187)} \right)$$



Step 2: Draw the root locus of G(z)

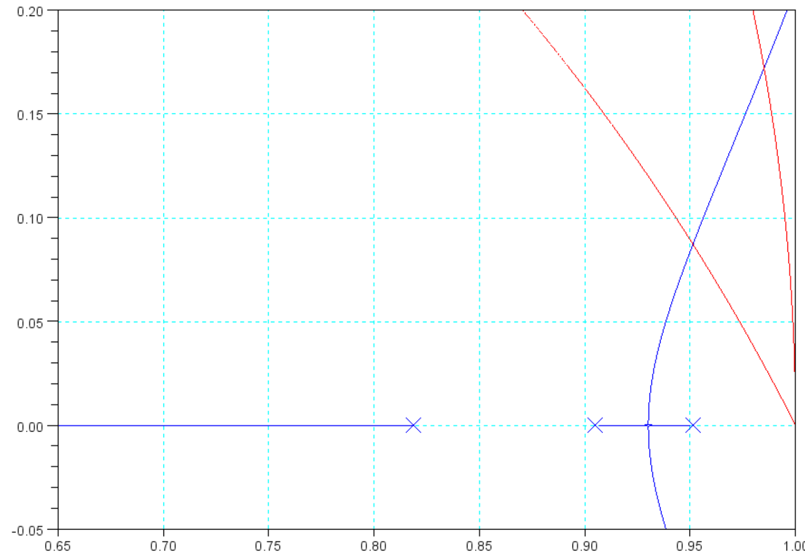
- Add the damping line

```
T = 0.01;
Gz = zpk(0, [0.9512, 0.9048, 0.8187],
0.0008413);
k = logspace(-2,2,1000)';
R = rlocus(G,k);

% draw the damping lines on this graph

hold on
s = [0:0.01:100] * (-1+j*2);
z = exp(s*T);
plot(real(z),imag(z),'r')

s = [0:0.01:100] * (j*1);
z = exp(s*T);
plot(real(z),imag(z),'r')
```



Step 3: Pick a spot on the root locus

a) No overshoot.

- This is the breakaway point

$$z = 0.9305$$

At this point

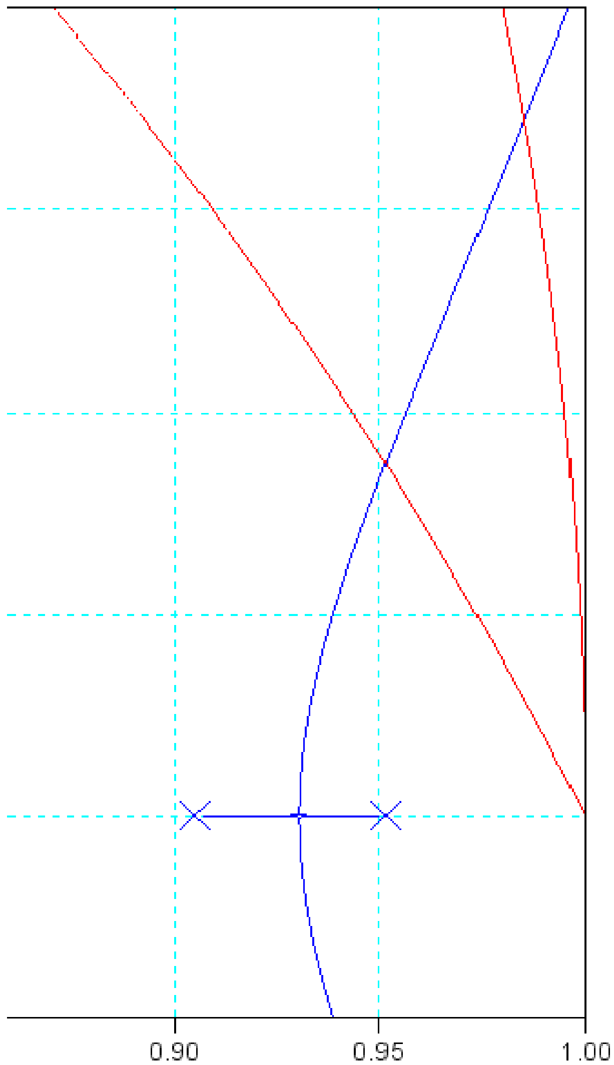
$$\left(\frac{0.0008413z}{(z-0.9512)(z-0.9048)(z-0.8187)} \right)_{z=0.9305} = 13.1620\angle 180^0$$

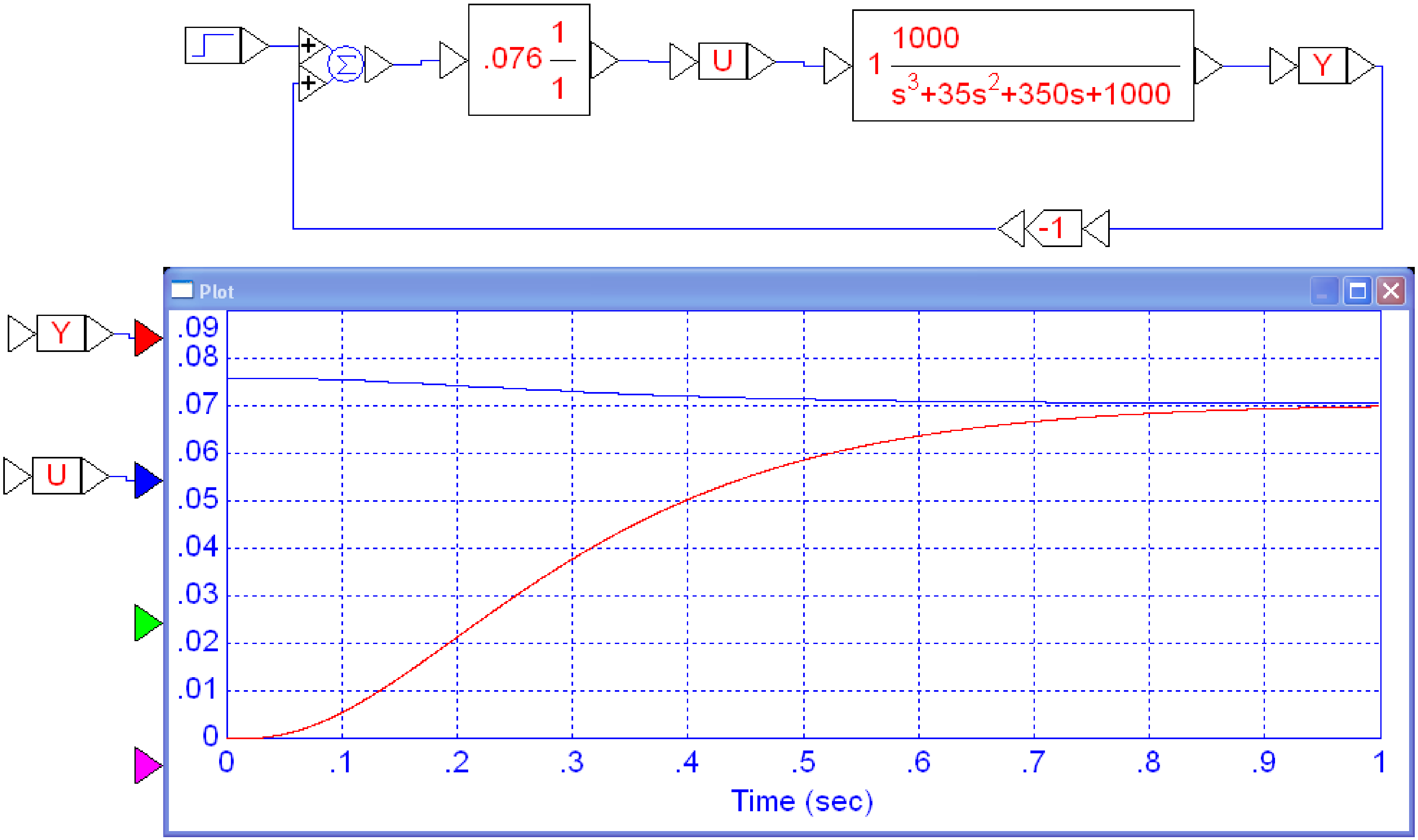
$$K = \frac{1}{13.1620} = 0.0760$$

This results in

$$t_{2\%} = \frac{\ln(0.02)}{\ln(0.9305)} = 54.3 \text{ samples} \quad (0.543 \text{ seconds})$$

$$Kp = 0.0760$$





Step Response with K(z) chosen to place the poles at the breakaway point (no overshoot).

b) 20% overshoot.

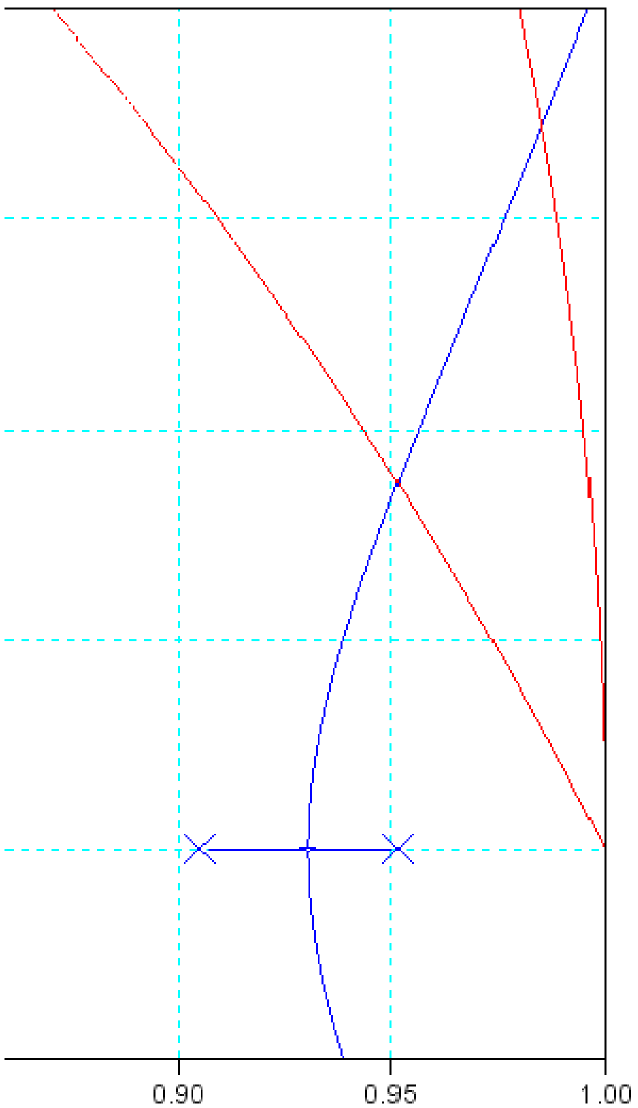
$z = 0.9513 + j0.0873$

At this point

$$\left(\frac{0.0008413z}{(z-0.9512)(z-0.9048)(z-0.8187)}\right)_{z=0.9513+j0.0873} = 0.5867\angle 180^0$$

K(z) is then

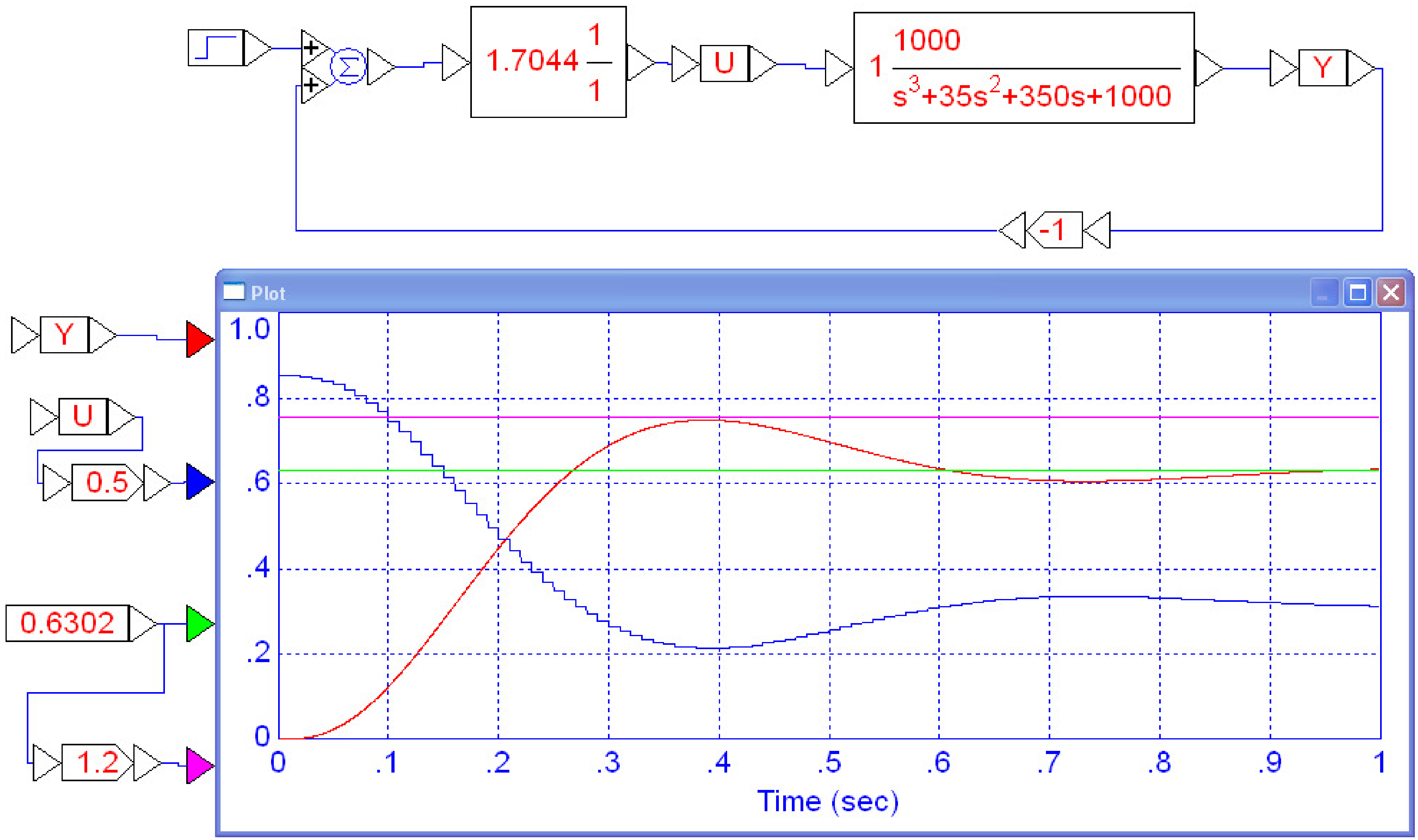
$$K = \frac{1}{0.5867} = 1.7044$$



This results in

$$t_{2\%} = \frac{\ln(0.02)}{\ln(|0.9513+j0.0873|)} = 85.52 \text{ samples}$$

Kp = 1.7044



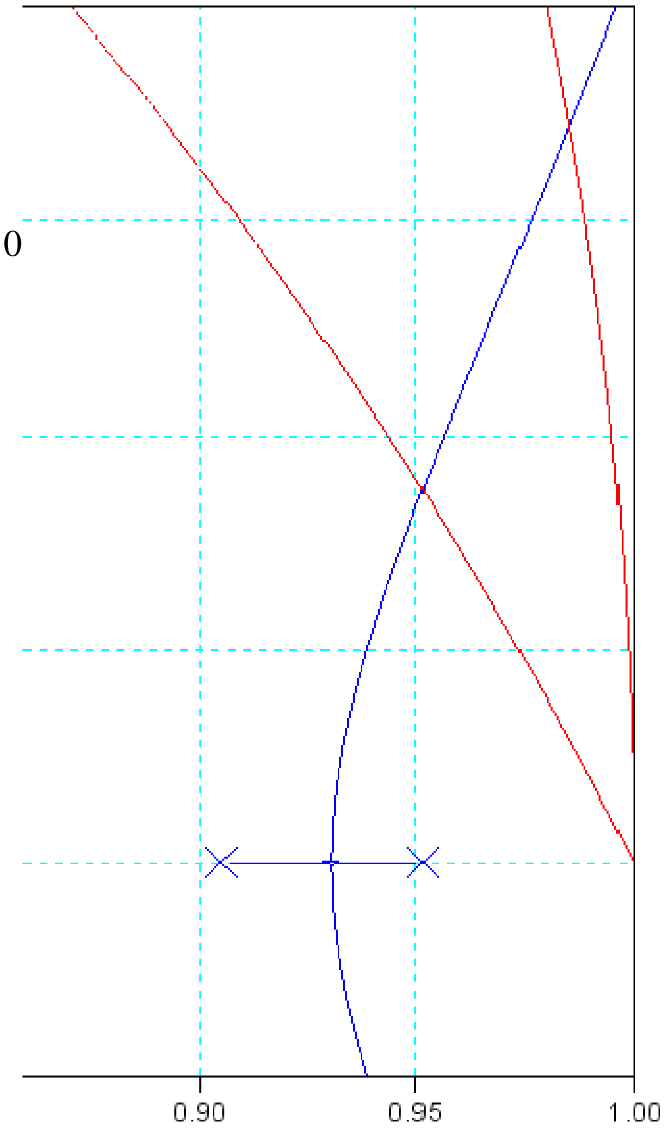
Step Response with K(z) chosen for 20% overshoot

c) Max Gain for Stability (the jw crossing).

$$z = 0.9850 + j0.1723$$

At this point

$$\left(\frac{0.0008413z}{(z-0.9512)(z-0.9048)(z-0.8187)} \right)_{z=0.9850+j0.1723} = 0.1053\angle 180^0$$



meaning

$$K(z) = \frac{1}{0.1053} = 9.5008$$

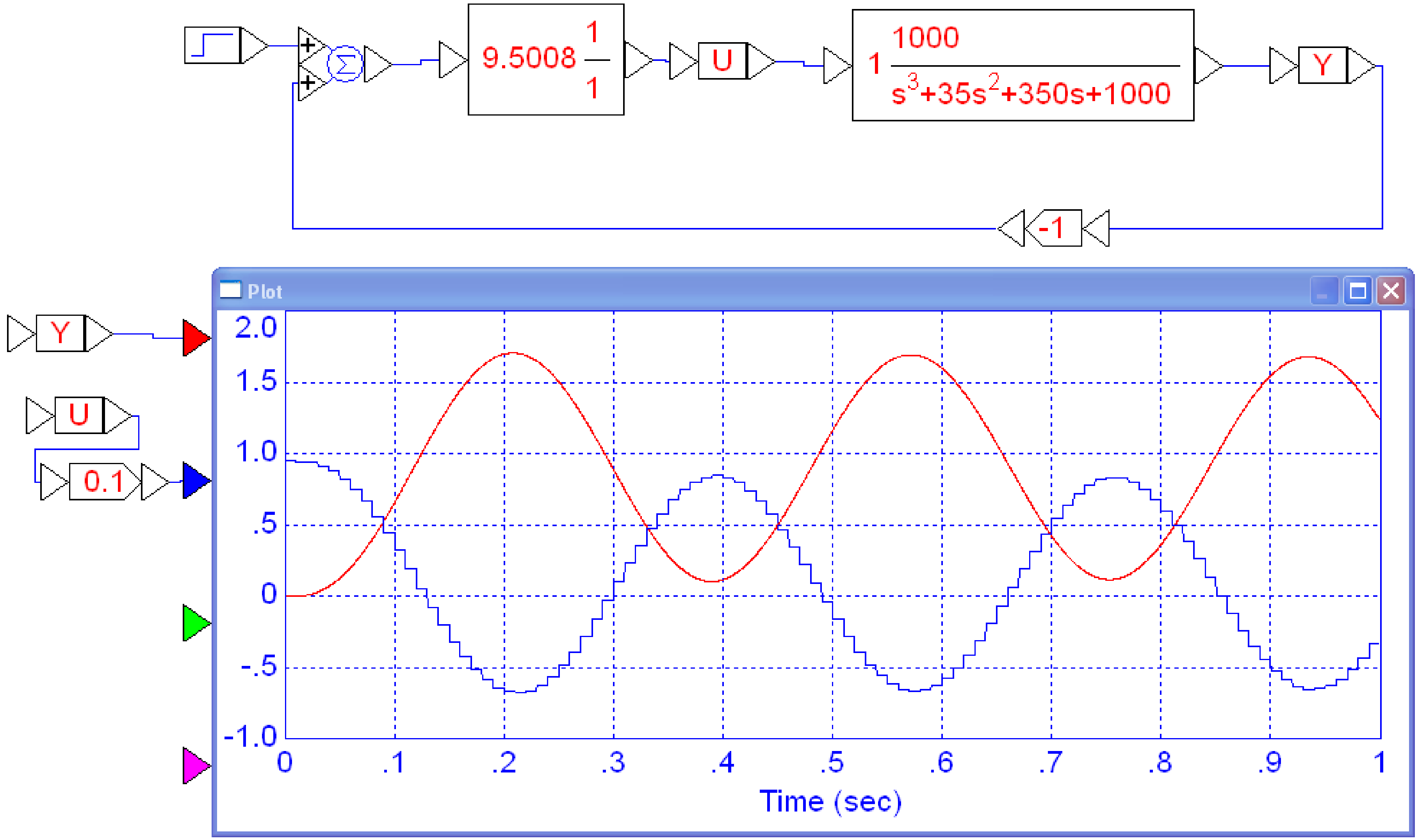
The frequency of oscillation is

$$\angle z = 9.9215^0$$

$$\text{period} = \frac{360^0}{9.9215^0} = 36.28 \text{ samples}$$

$$= 0.3628 \text{ seconds}$$

$$f = \frac{1}{period} = 2.75 \text{ Hz}$$



9.5008
1
1
U
1
1000
s^3+35s^2+350s+1000
Y
-1
Plot
Y
U
0.1
2.0
1.5
1.0
.5
0
-.5
-1.0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
Time (sec)

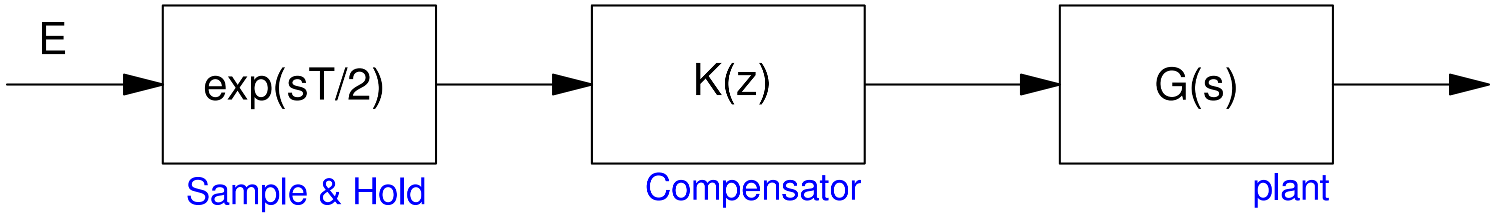
Alternate Method:

Find the spot on the damping line where the angles add up to 180 degrees

$$G \cdot K = 1\angle 180^0$$

When you have a digital compensator, you really have three terms:

G(s) * K(z) * sample and hold



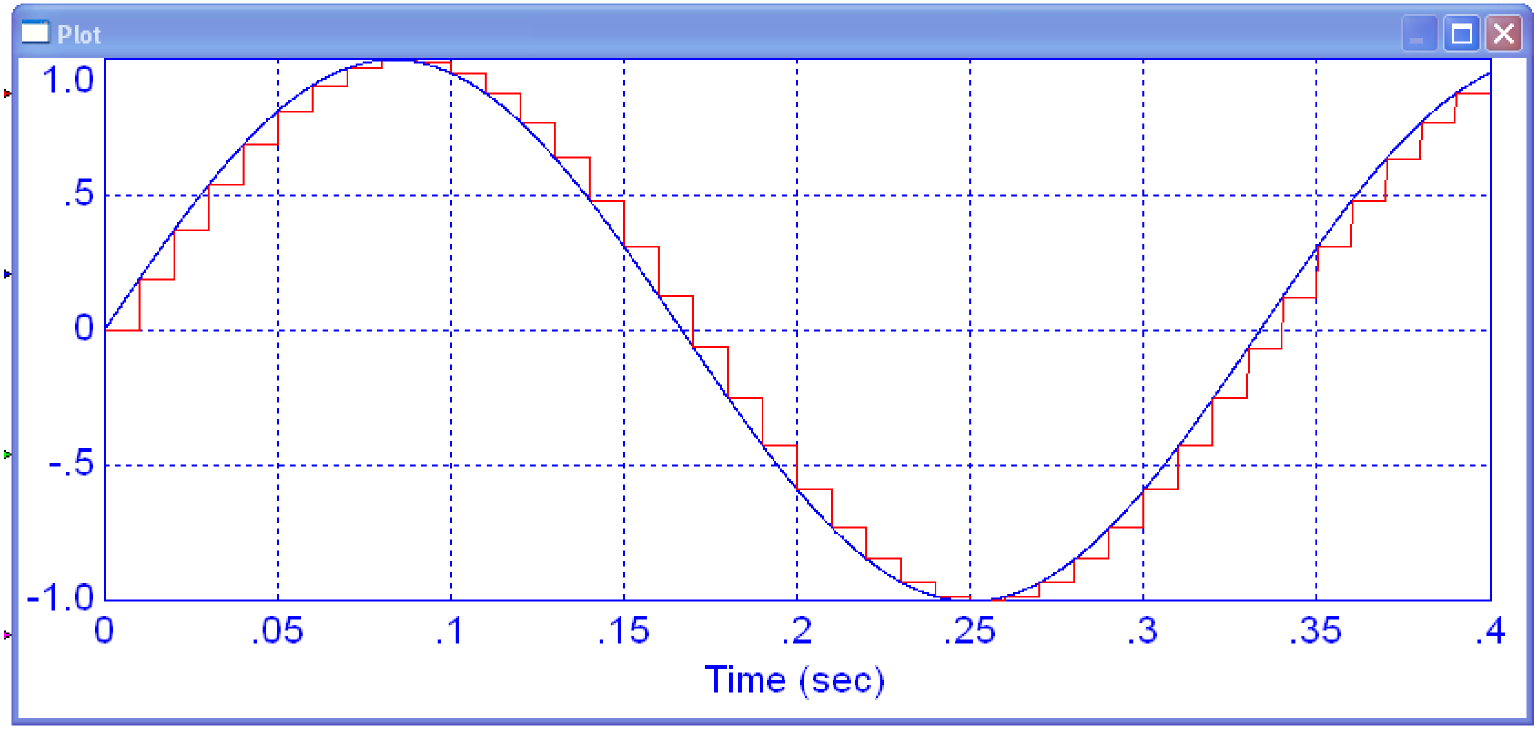
Matlab can analyze each of these

- You don't have to do s to z conversions

Modeling the Sample & Hold

- Sample and hold adds a 1/2 sample delay

$$H(s) \approx \exp\left(\frac{-sT}{2}\right)$$



3Hz sine wave (blue) sampled at 10ms (red) results in a 3Hz sine wave delayed by 1/2 of a sample (5ms)

Open-Loop System is then

$$G(s) \cdot K(z) \cdot \exp\left(\frac{-sT}{2}\right)$$

Find the point on the damping line where the phase is 180 degrees

$$\angle\left(G(s) \cdot K(z) \cdot \exp\left(\frac{-sT}{2}\right)\right) = 180^0$$

Note: You're kind-of mixing planes with this approach. Since you only care about one point, however, you don't care. You just

- Guess the point, s
- Compute the corresponding point in the z-plane as $z = e^{sT}$
- Evaluate the above function, and
- Repeat until the angles add up to 180 degrees

Example: Find the gain, k, that results in 20% overshoot in the step response.

Guess #1: s = -5 + j10

```
T = 0.01;
s = 5*( -1 + j*2);
z = exp(s*T);

1000 / ( (s+5)*(s+10)*(s+20))  *  exp(-s*T/2)

  - 0.5009862 + 0.0882455i
```

The angle isn't zero (the complex part is non-zero)

- Try a different s, such as 10% smaller

```
s = s*0.9;
z = exp(s*T);
1000 / ( (s+5)*(s+10)*(s+20))  *  exp(-s*T/2)

  - 0.5110062 + 0.0796359i
```

Keep going until the complex part is zero

- Angle is 180 degrees
- time pases.....

```
s = 0.9999*s;
1000 / ( (s+5)*(s+10)*(s+20))  *  exp(-s*T/2)

ans =  -0.5853 - 0.0000i
```

Close enough. This gives

```
>> k = 1/abs(ans)

k =    1.7085

>> s

s =  -4.5760 + 9.1519i

>> z

z =   0.9465 + 0.0950i
```

Note:

- The numerical method gives almost the same answer as root locus
- It's easier and is actually more accurate

    no s to z conversions

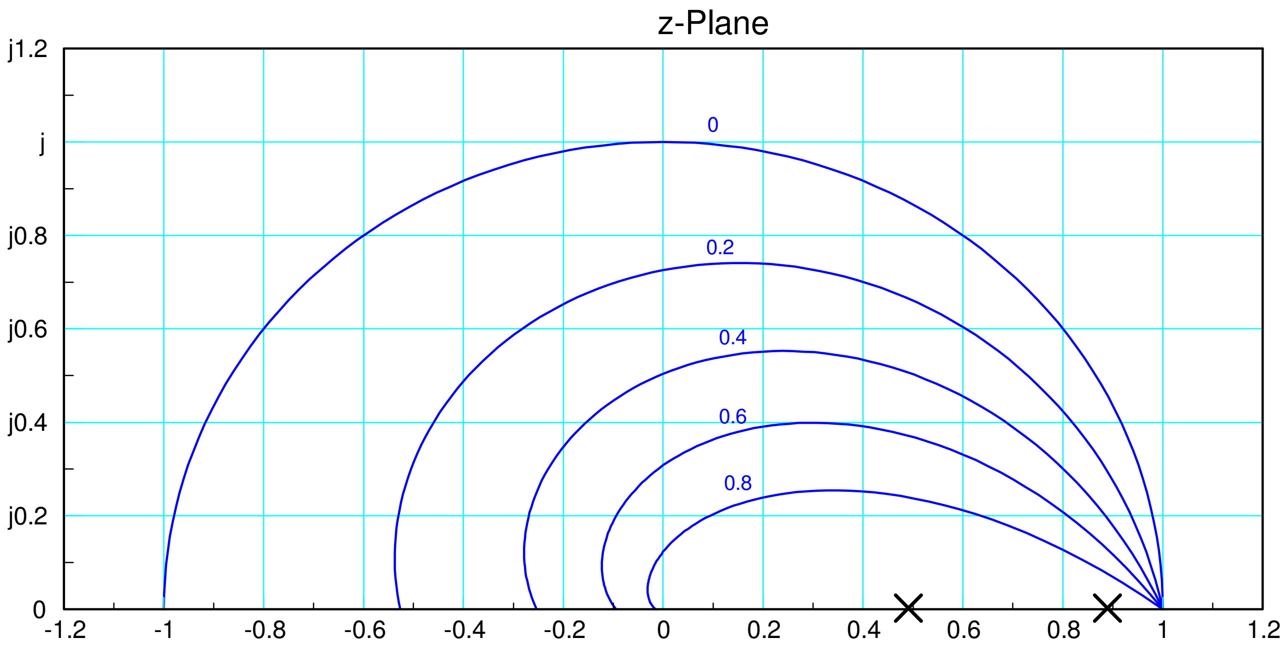
	Root Locus	Numerical Approach
k	1.7044	1.7085
s	-	-4.5760 + 9.1519i
z	0.9513 + j0.0873	0.9465 + 0.0950i

Handout: Sketch the root locus of

$$G(z) = \left(\frac{0.2}{(z-0.9)(z-0.5)}\right)$$

Find k for

- The breakaway point
- A damping ratio of 0.4
- The maximum gain for stability



Summary

- Root locus works in the s-plane
- Root locus works in the z-plane

The only difference is how you interpret the result:

$$T_s = \left(\frac{\ln(0.02)}{\ln(|z|)} \right)$$

$$period = \left(\frac{360^0}{\angle(z)} \right) \cdot T \quad \text{seconds}$$

$$z = e^{sT}$$