Lead Compensators using Root Locus

ECE 461/661 Controls Systems

Jake Glower - Lecture #23

Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

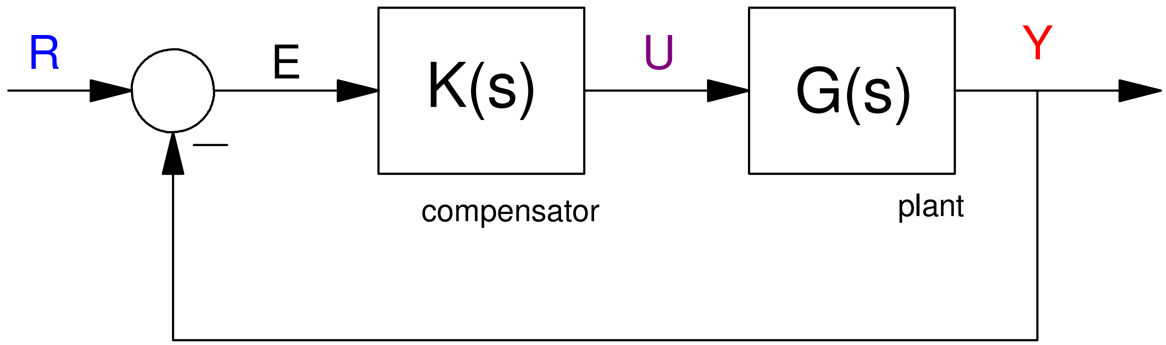
Introduction

Goal: Design a compensator, K(s), to give

- Good tracking (error constants are proportional to k) and
- A fast response

Meaning...

- Make K(s) large (faster system, better tracking),
- But not too large (stable and not too much oveshoot).



Lead Compensator Design

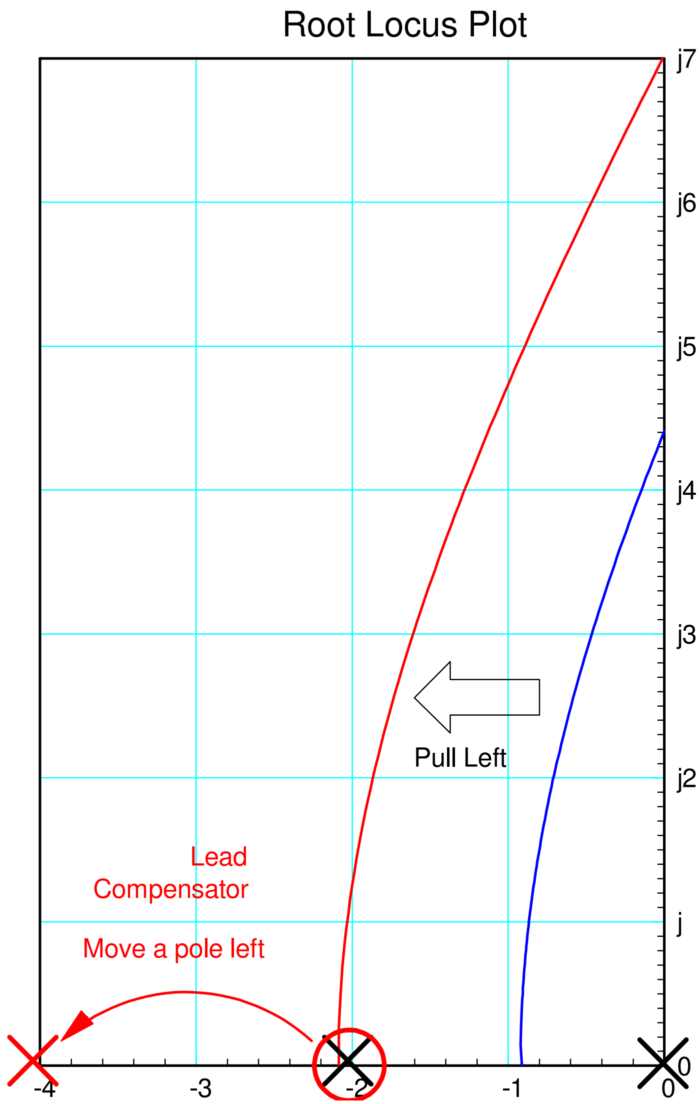
- $K(s) = k\left(\frac{s+a}{s+b}\right)$ $\quad b > a$

Add a zero to cancel a pole

- Get rid of the one that's causing problems

Replace it with a faster pole

- Avoids differentiation (noise amplifier)
- Pulls the root locus left
- Speeds up the system



Which Pole do you Cancel?

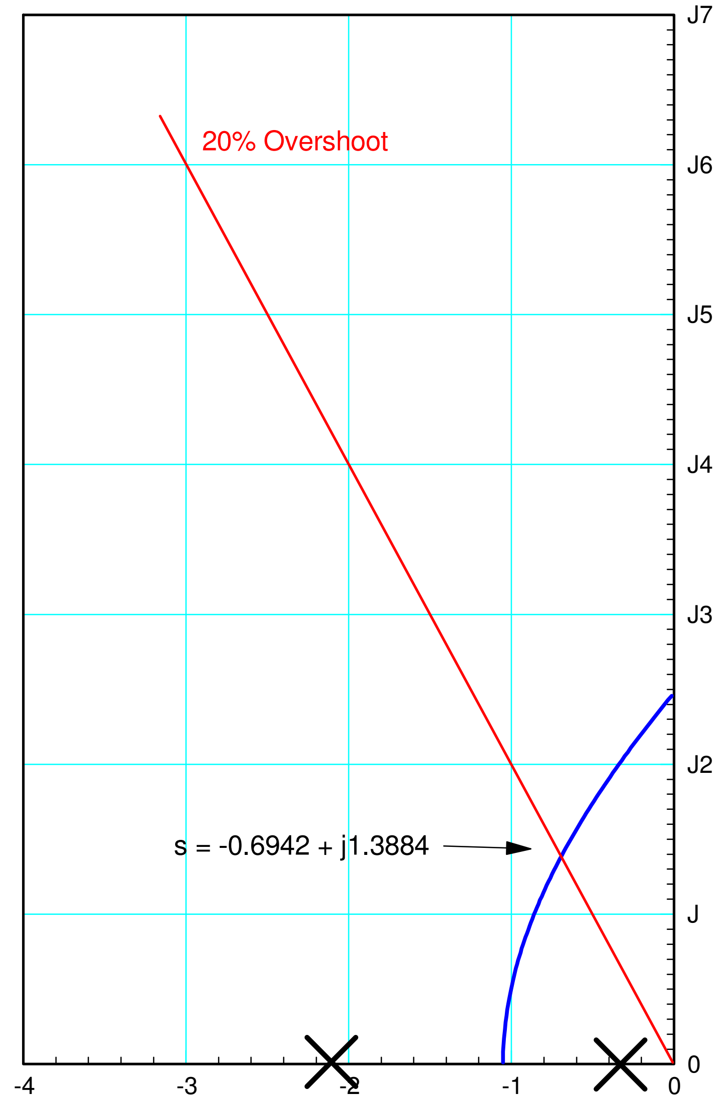
Example: 5th Order System

$$G(s) = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)}\right)$$

Cancel Nothing (Gain Compensation)

For 20% Overshoot

- s = -0.6942 + j1.3884
- K(s) = 5.5117
- Kp = 3.4412
- Ts = 5.76 seconds



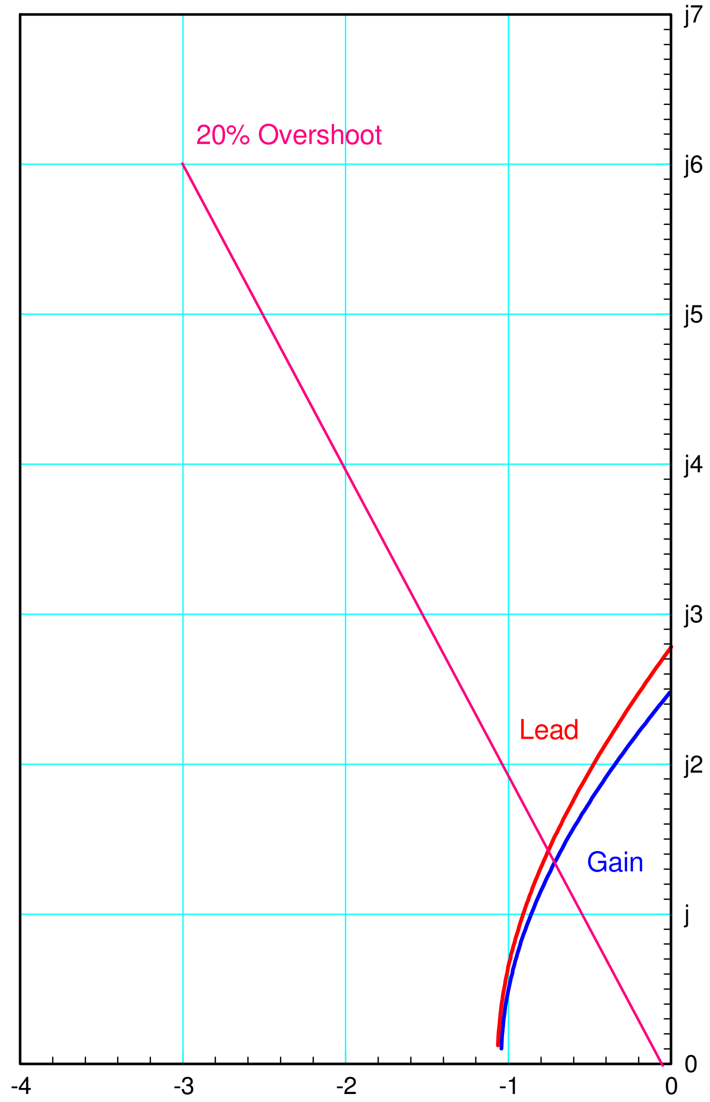
Cancel the Fastest Pole

$$G(s) = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)}\right)$$

$$K(s) = k\left(\frac{s+15.65}{s+156.5}\right)$$

Result

- Almost no change
- The fast pole isn't the problem



Cancel slowest pole

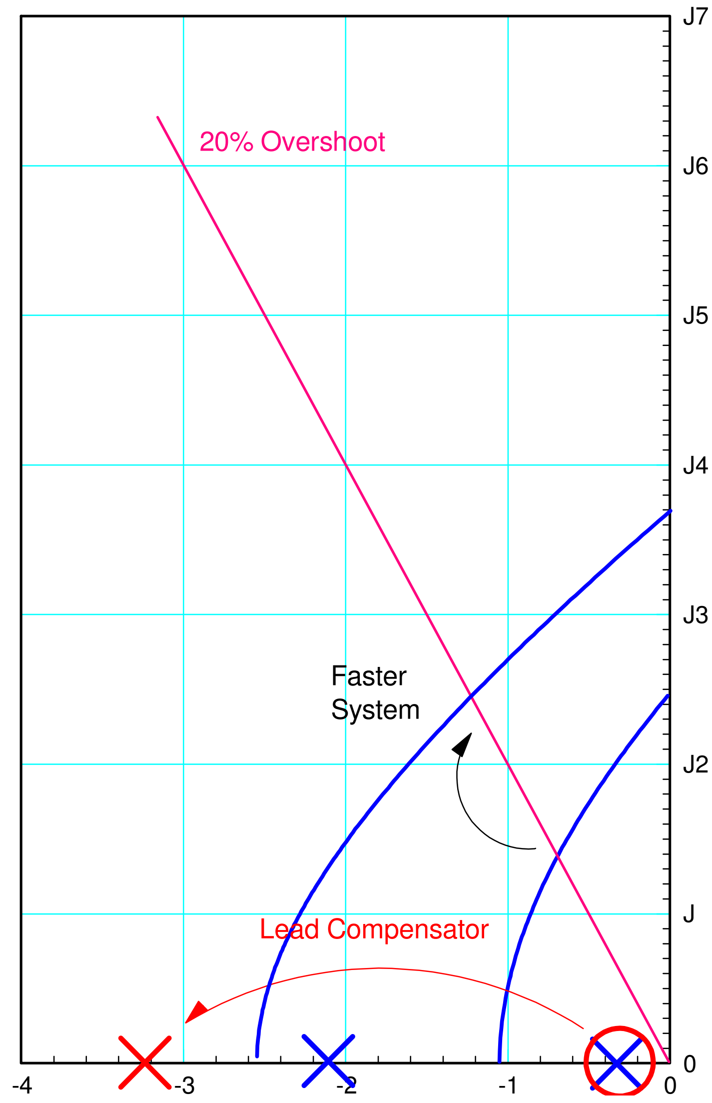
$$G(s) = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)} \right)$$

$$K(s) = k\left(\frac{s+0.3242}{s+3.242} \right)$$

$$GK = \left(\frac{361.2378k}{(s+15.65)(s+10.1)(s+5.439)(s+2.081)(s+3.234)} \right)$$

For 20% Overshoot

- s = -1.2531 + j2.5062
- $K(s) = 15.30\left(\frac{s+0.3242}{s+3.242} \right)$
- Kp = 0.9554 (worse)
- Ts = 3.19 sec (better)

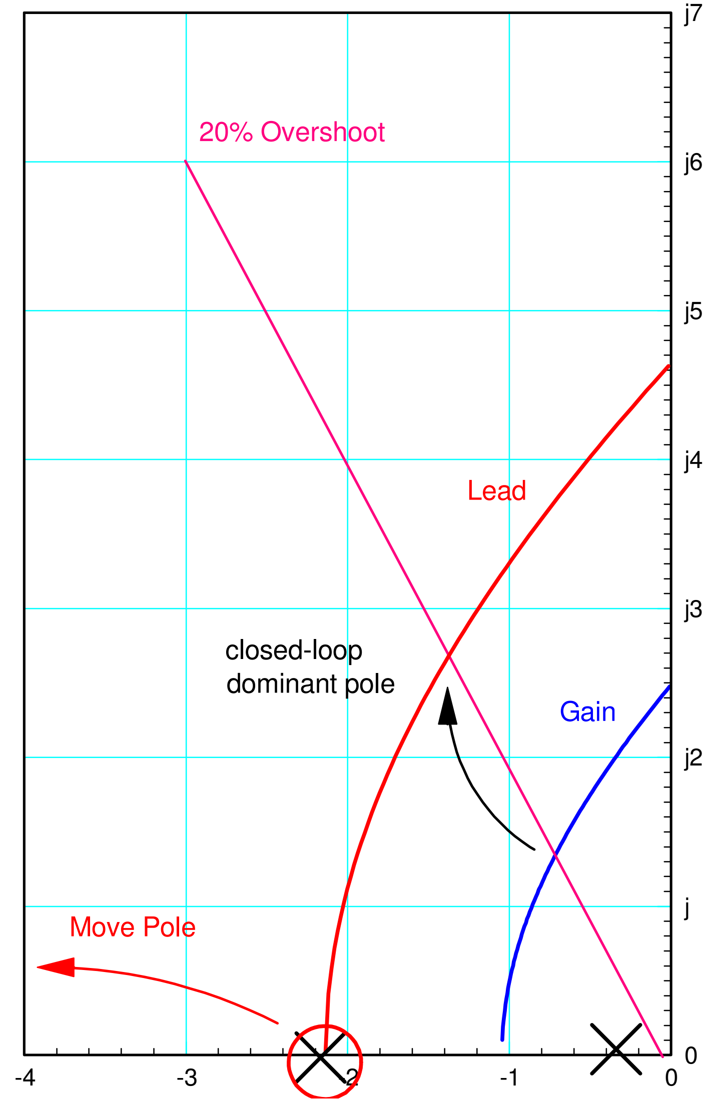


Cancel the 2nd Slowest Pole

- Keep the pole at s = -0.3234
  - Like a pole at s = 0, reduces the steady-state error
- Cancel the pole at s =

$$G(s) = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)}\right)$$

$$K(s) = k\left(\frac{2.081}{s+20.81}\right)$$



For 20% Overshoot

- s = -1.3501 + j2.7002
- $K(s) = 102.59\left(\frac{s+2.081}{s+20.81}\right)$
- Kp = 6.4052 (2x better)
- Ts = 2.9627 (better)

Computations

$$K(s) = k\left(\frac{s+2.081}{s+20.81}\right)$$

$$GK = \left(\frac{361.2378k}{(s+20.81)(s+15.65)(s+10.1)(s+5.439)(s+0.3234)}\right)$$

$$\left(\frac{361.2378}{(s+20.81)(s+15.65)(s+10.1)(s+5.439)(s+0.3234)}\right)_{s=-1.3501+j2.7002} = 0.0097\angle 180^0$$

$$k = \frac{1}{0.0097} = 102.59$$

$$K(s) = 102.59\left(\frac{s+2.081}{s+20.81}\right)$$

$$T_{2\%} = \frac{4}{1.3501} = 2.9627 \text{ seconds}$$

$$K_p = (G(s)\cdot K(s))_{s=0} = 6.4052$$

Results:

Note that canceling the pole at -2.081 resulted in

- A slightly faster system
- With a much larger error constant, Kp, meaning better tracking
- As well as a much larger value of U at t=0

Lead Compensation K(s) = k (s+a) / (s+10a)				
K(s)	Closed-Loop Dominant Pole(s)	U at t=0 K(s) as s -> infinity	Kp	2% Settling Time seconds
5.5117	s = -0.6942 + j1.3884	5.5117	3.4412	5.76
$15.30\left(\frac{s+0.3242}{s+3.242}\right)$	s = -1.2531 + j2.5062	15.30	0.9554	3.19
$102.59\left(\frac{s+2.081}{s+20.81}\right)$	s = -1.3501 + j2.7002	102.59	6.4052	2.96

Lead Compensator Design

Keep one pole at or near s=0.

- Pick the zero of the lead compensator to cancel the next slowest (stable) pole.
- Keeps Kp (or Kv) large

Pick the pole 3 to 10 times larger than the zero

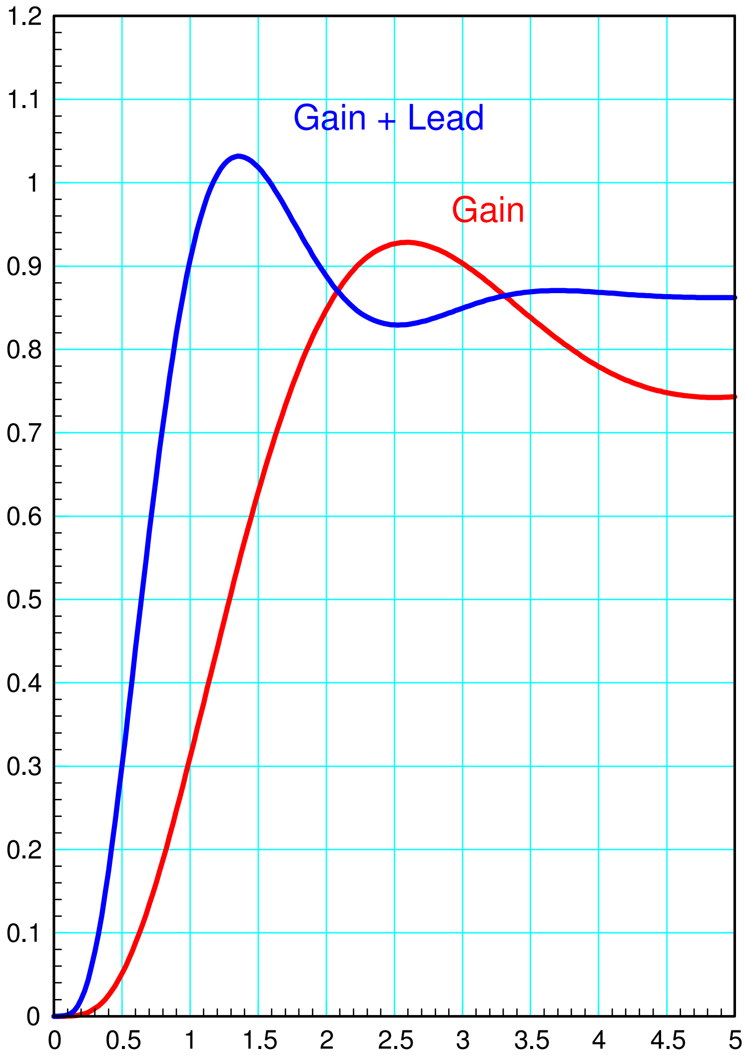
- Speeds up the system
- There is *some* limit on how much input you can apply
- Assume 3x to 10x faster

Resulting Step Response

```
>> G = zpk([],[-0.3243,-2.081,-5.439,-10.1,-15.65], 361.2378);
>> K = zpk(-2.081, -20.81, 101.61)

101.61 (s+2.081)
----------------
   (s+20.81)

>> t = [0:0.01:5]';
>> Gcl = minreal(G*K / (1+G*K));
>> y = step(Gcl, t);
>> plot(t,y);
```



Gain + Lead is

- Faster
  - root locus is further left
- With better tracking
  - Less steady-state error
  - Larger error constant

Lead Compensator Circuit: $K(s) = 101.61\left(\frac{s+2.081}{s+20.81}\right)$

3 constraints, 4 degrees of freedom

- Assume R2 = 1M

High frequency gain ($s \to \infty$)

$$K(s) = 101.61 = \frac{R_2}{R_a}$$

Ra = 9.84k

DC Gain ($s \to 0$)

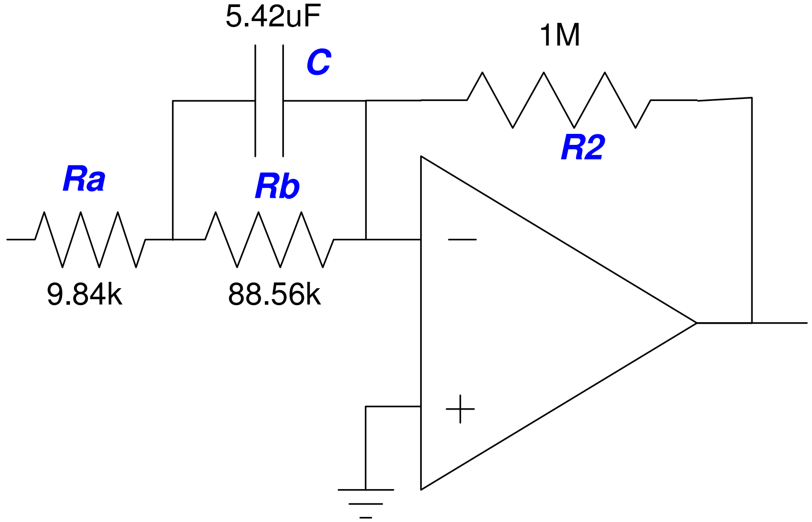
$$K(s) = 10.161 = \frac{R_2}{R_a + R_b}$$

Rb = 88.56k

Zero:

$$\frac{1}{R_b C} = 2.081$$

C = 5.42uF



Lead Compensator: Software

$$K(s) = 101.61\left(\frac{s+2.081}{s+20.81}\right) = 101.61\left(1 - \frac{18.729}{s+20.81}\right)$$

Add a dummy state, Z, which is

```
Z = 0;

R = 100;

while(t < 100)

   E = R - V(10);
   dZ = -20.81*Z + 18.729*E;

   V0 = 101.61 * ( E - Z );

   dV(1)  = 10*V0   - 20.1*V(1) + 10*V(2);
   dV(2)  = 10*V(1) - 20.1*V(2) + 10*V(3);
   dV(3)  = 10*V(2) - 20.1*V(3) + 10*V(4);
   :
   :
```

Lead Compensators x 2

- If one lead compensator is good, why not two?

Keep the slowest pole (s = -0.3234)

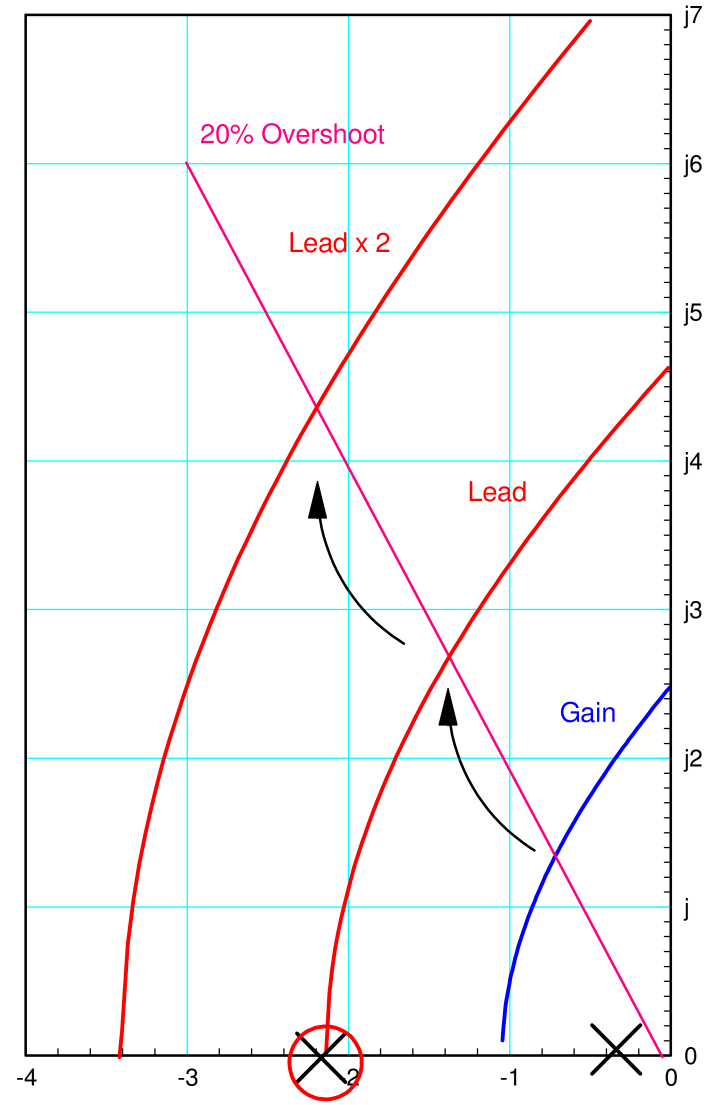
Cancel the next two slowest poles

$$K(s) = k\left(\frac{s+2.081}{s+20.81}\right)\left(\frac{s+5.439}{s+53.39}\right)$$

$$GK = \left(\frac{361.2378k}{(s+15.65)(s+10.1)(s+54.39)(s+20.81)(s+0.3234)}\right)$$

Result:

- s = -2.2463 + j4.4925
- $K(s) = 1729.60\left(\frac{s+2.081}{s+20.81}\right)\left(\frac{s+5.439}{s+53.39}\right)$

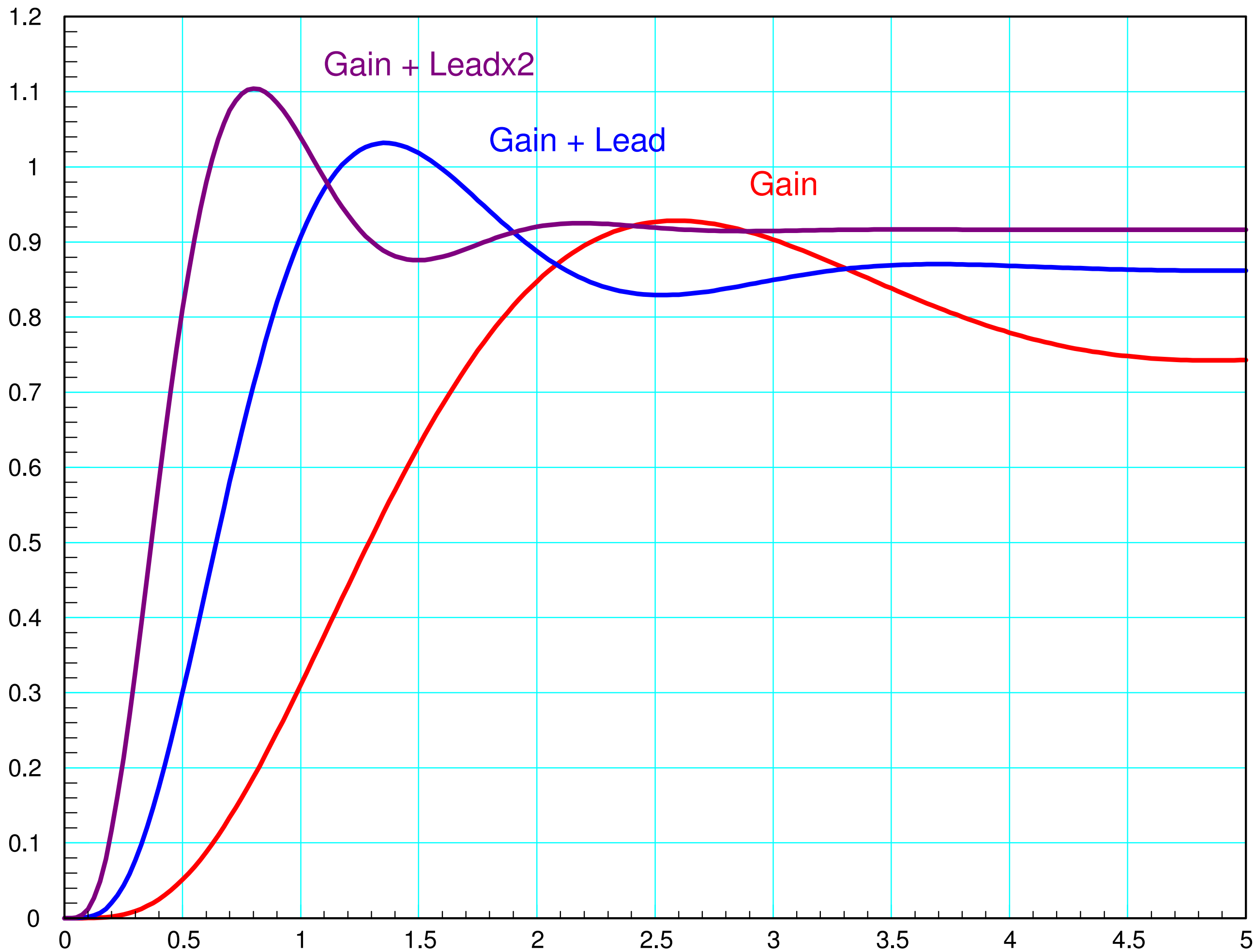


Result

- Each lead compensator speeds up the sytem
- The input at t = 0 increses 31x though...

Lead Compensation K(s) = k (s+a) / (s+10a)				
K(s)	Closed-Loop Dominant Pole(s)	U at t = 0 K(s) as s -> infinity	Kp	$T_{2\%}$ seconds
5.5117	s = -0.6942 + j1.3884	5.5117	3.4412	5.76
$15.30\left(\frac{s+0.3242}{s+3.242}\right)$	s = -1.2531 + j2.5062	15.3	0.9554	3.19
$102.59\left(\frac{s+2.081}{s+20.81}\right)$	s = -1.3501 + j2.7002	102.59	6.4052	2.96
$1729.60\left(\frac{s+2.081}{s+20.81}\right)\left(\frac{s+5.439}{s+53.39}\right)$	s = -2.2463 + j4.4925	1729.60	10.7987	1.78

Closed-Loop Response (Gain, Lead, Lead x 2):

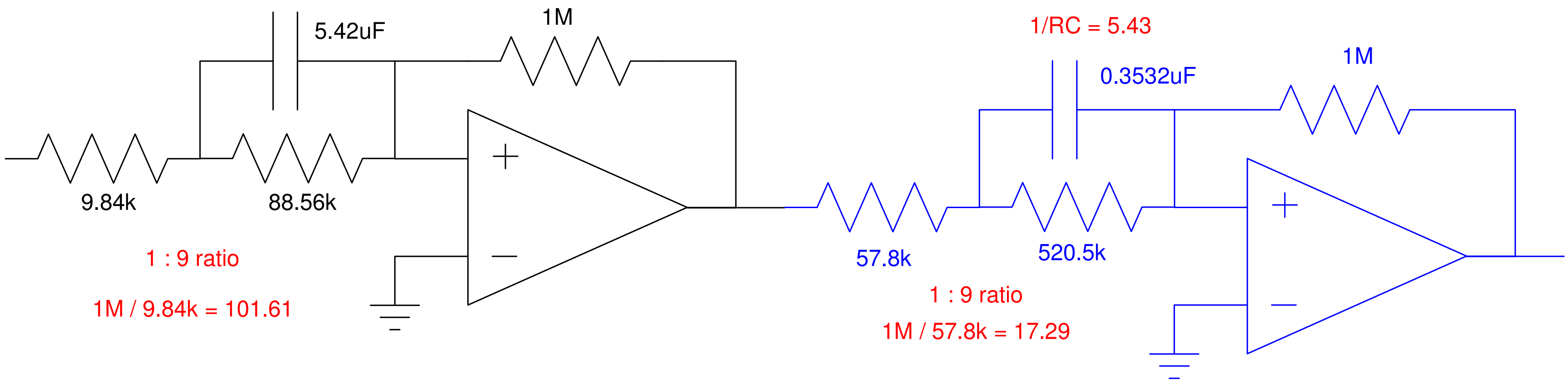


Step Response for the Gain, Lead, and Two Lead Compensated Systems

Lead Circuit

$$K(s) = 1729.60\left(\frac{s+2.081}{s+20.81}\right)\left(\frac{s+5.439}{s+54.39}\right)$$

$$K(s) = \left(101.61\frac{s+2.081}{s+20.81}\right)\left(17.29\frac{s+5.439}{s+54.39}\right)$$

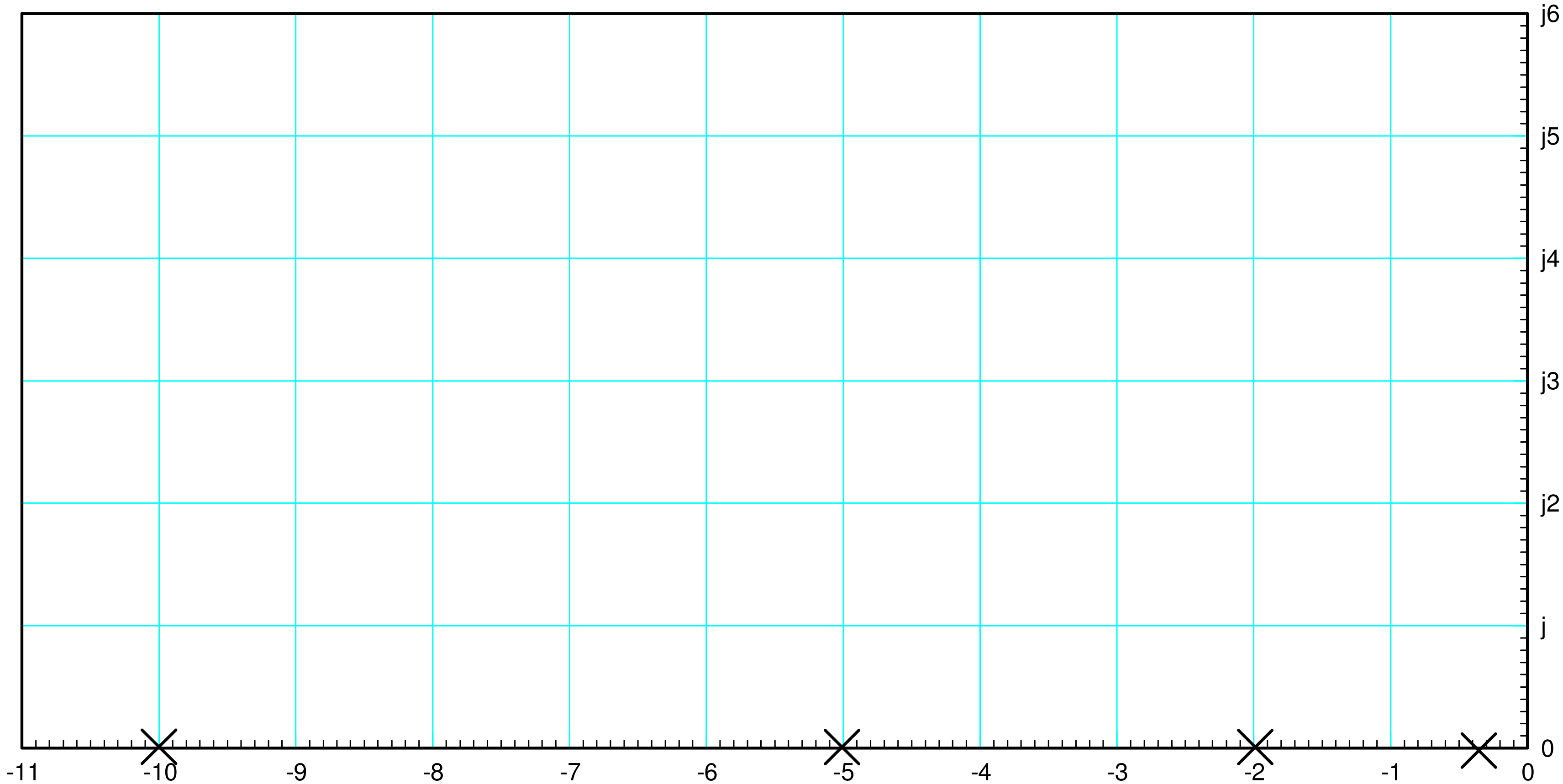


Circuit to implement a 2-stage lead compensator

Handout: Design a lead compensator for the following system

$$G(s) = \left(\frac{200}{(s+0.3)(s+2)(s+5)(s+10)}\right)$$

so that the damping ratio is 0.707 (45 degrees)



Summary:

Lead compensators improve the closed-loop dynamics by pulling the root locus left

Procedure:

- Keep the pole closest to s=0
- Cancel the next slowest stable pole
- Replace it with a pole 3x to 10x faster