Root Locus with Repeated and Complex Poles & Zeros

ECE 461/661 Controls Systems

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Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Root Locus with Repeated Poles:

Treat these as N poles separated by ε

$$\left(\frac{100}{s^2(s+5)(s+20)}\right) \approx \left(\frac{100}{(s+0.01)(s-0.01)(s+5)(s+20)}\right)$$

From the lecture on error constants

- One pole at s=0 is good: it causes the system to have no error for a step input
- Two poles at s = 0 is better: it causes the system to have no error for a step or ramp input

So, why not add a bunch of poles at s = 0?

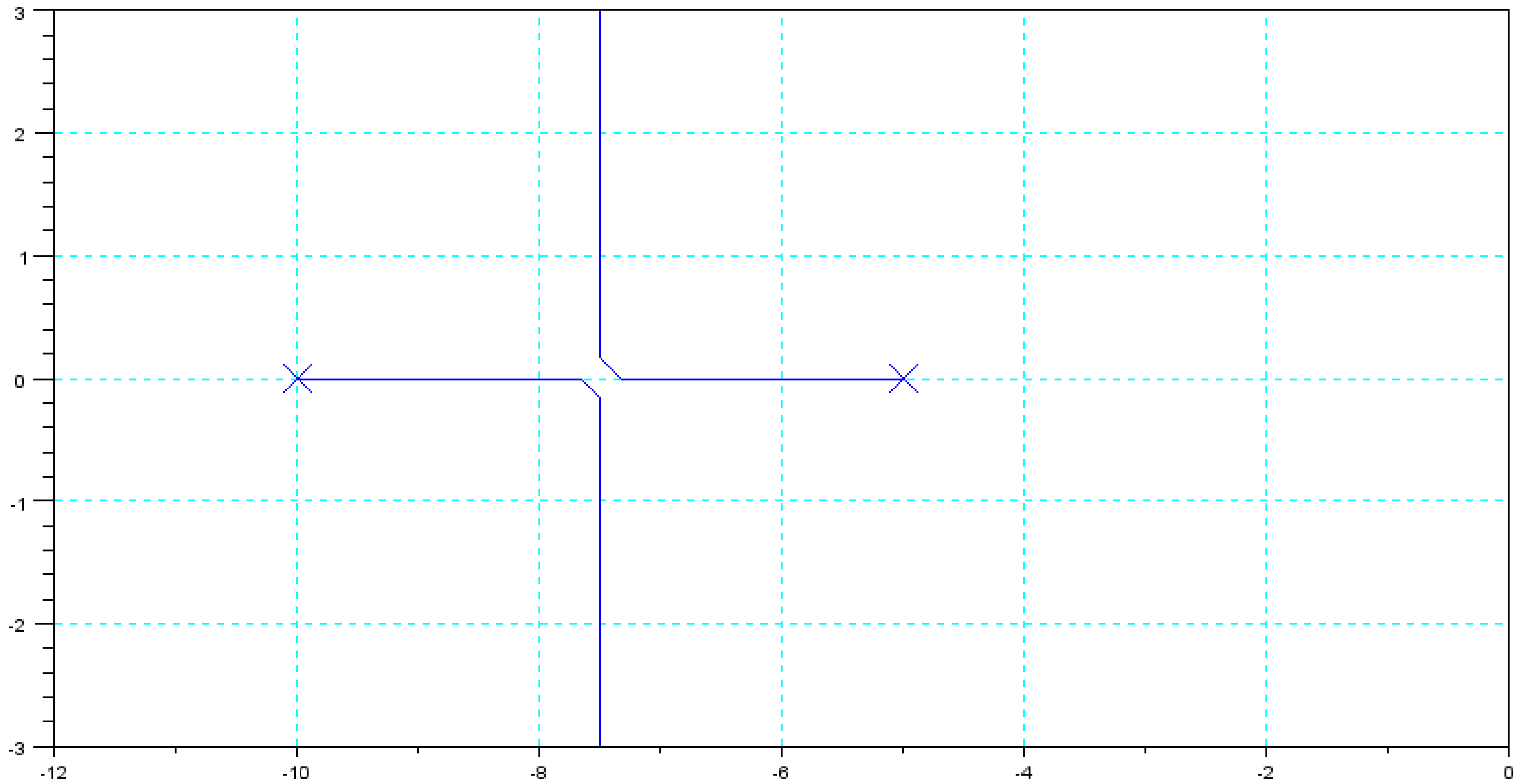
Why not a Type-20 system?

Problem shows up on root locus plots

- $G_0(s) = \left(\frac{10}{(s+5)(s+10)}\right)$ *a type-0 system*
- $G_1(s) = \left(\frac{10}{s(s+5)(s+10)}\right)$ *a type-1 system*
- $G_2(s) = \left(\frac{10}{s^2(s+5)(s+10)}\right)$ *a type-2 system*
- $G_3(s) = \left(\frac{10}{s^3(s+5)(s+10)}\right)$ *a type-3 system*

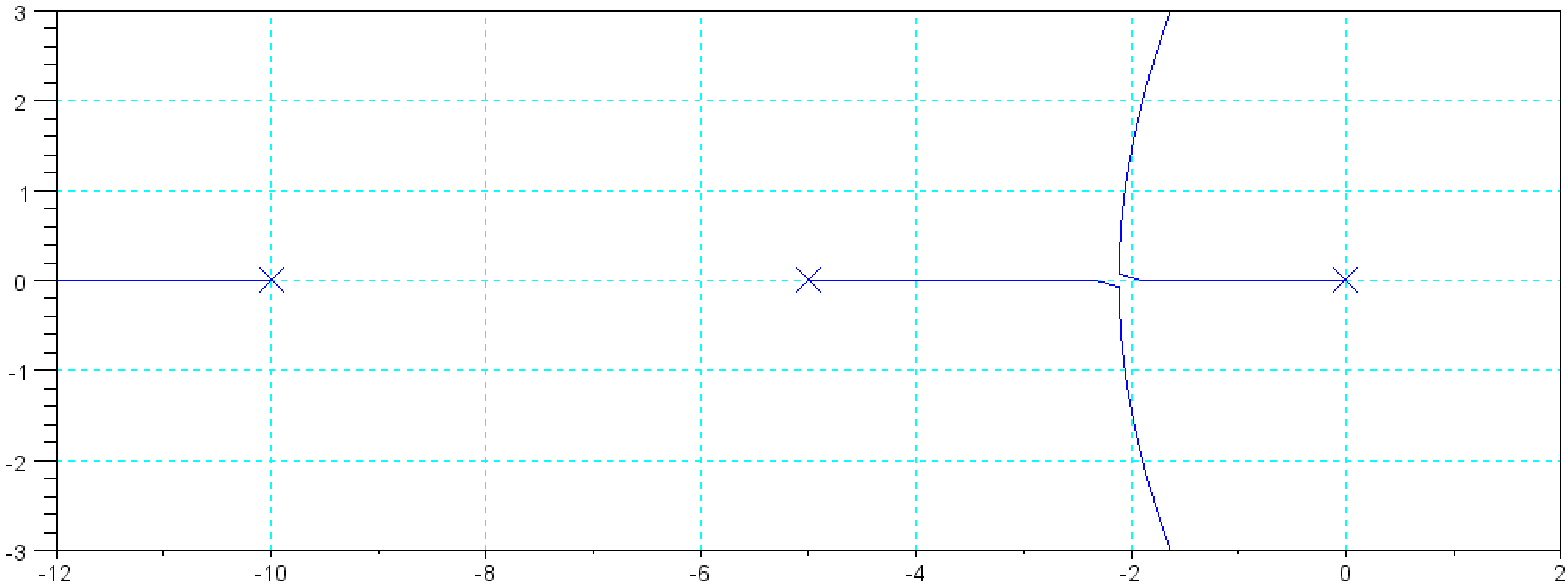
Type-0 System: $G_0(s) = \left(\frac{10}{(s+5)(s+10)} \right)$

- Stable for all k
- Easy to work with



Type 1 System: $G_1(s) = \left(\frac{10}{s(s+5)(s+10)}\right)$

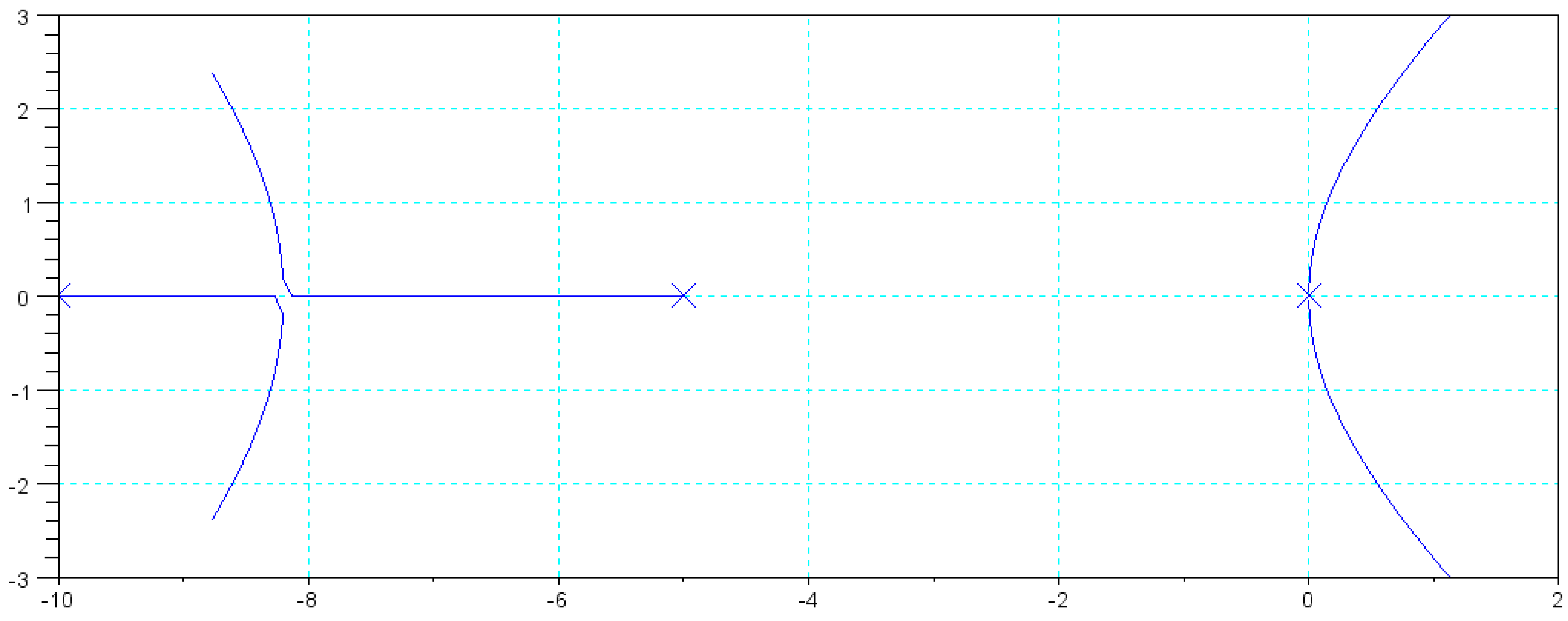
- Easy to stabilize: start with a small gain and increase
- Too much gain makes it go unstable



Root Locus for a Type-1 System

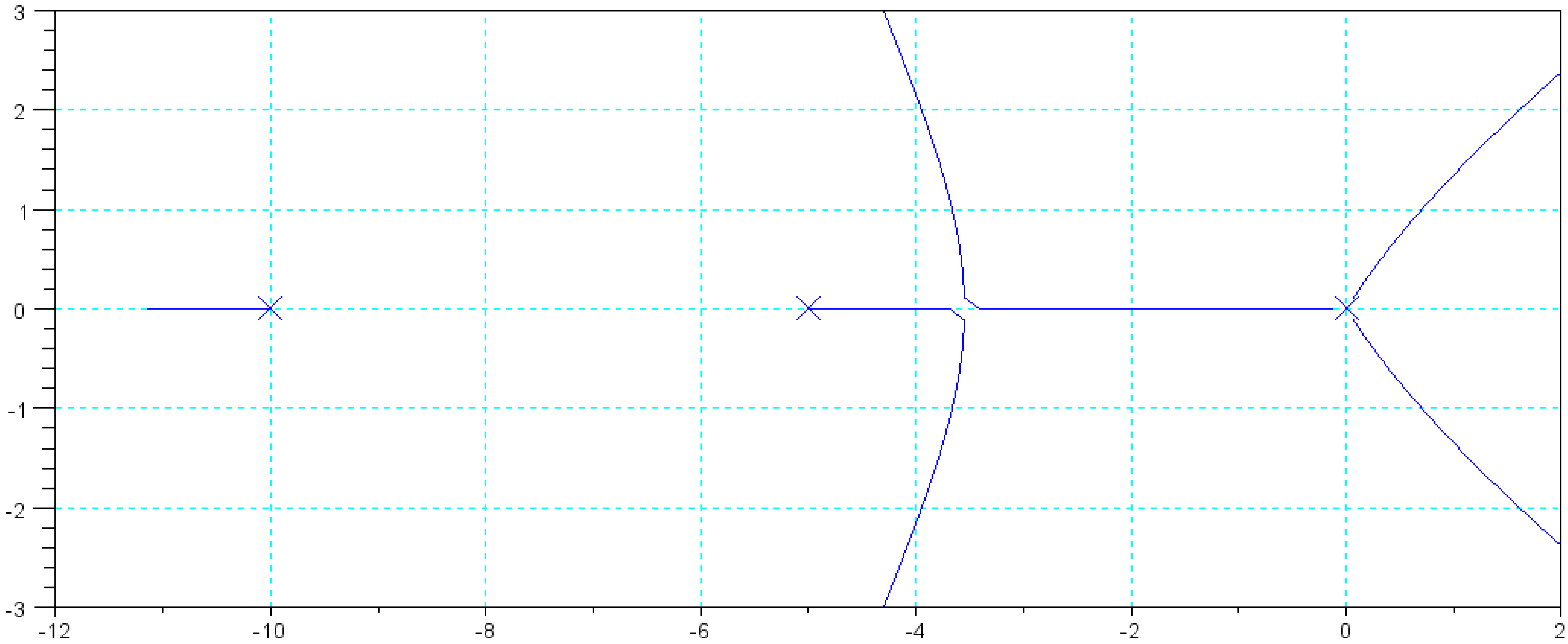
Type 2 System: $G_2(s) = \left(\frac{10}{s^2(s+5)(s+10)}\right)$

- Small k: oscillates
- Larger k: unstable and oscillates
- Need to add zeros to attract the root locus into the left half plane



Type 3 Systems: $G_3(s) = \left(\dfrac{10}{s^3(s+5)(s+10)}\right)$

- Problems get worse
- Becomes explosively unstable
- Very difficult to pull root locus into the left-half plane



Moral:

- Don't try to make the system Type-2 or higher
  - Solves a made-up problem (shooting down German bombers over London)
  - Makes the system *very* hard to stabilize
- *Do* strive for a type-1 system.
  - Easy to stabilize
  - No error for a step input
  - Step inputs are really common (constant set point)

Complex Poles and Departure Angles

How to draw a root locus with comples poles?

- What angle does the root locus leave the pole?
- *Departure angle:* Angle from a complex pole
- *Approach angle:* Angle from a complex zero

Answer: The angle that results in

$$angle(GK) = 180^0$$

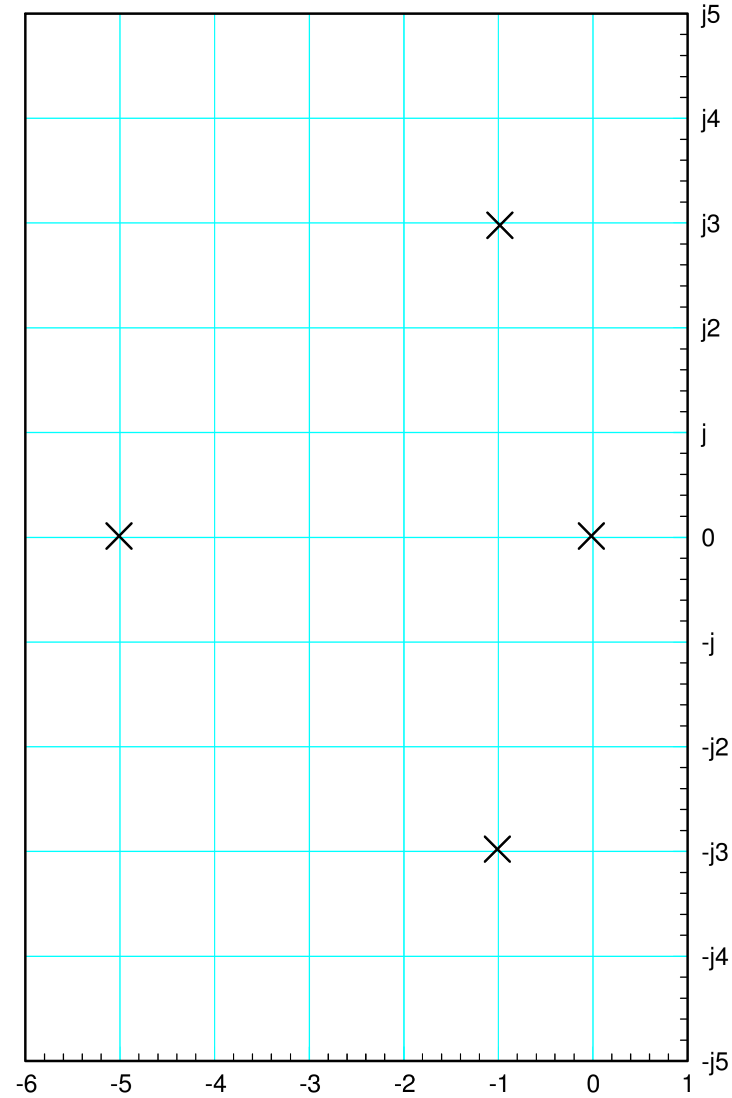
or

$$\Sigma\,(\text{angles from poles to s}) - \Sigma\,(\text{angles from zeros to s}) = 180^0$$

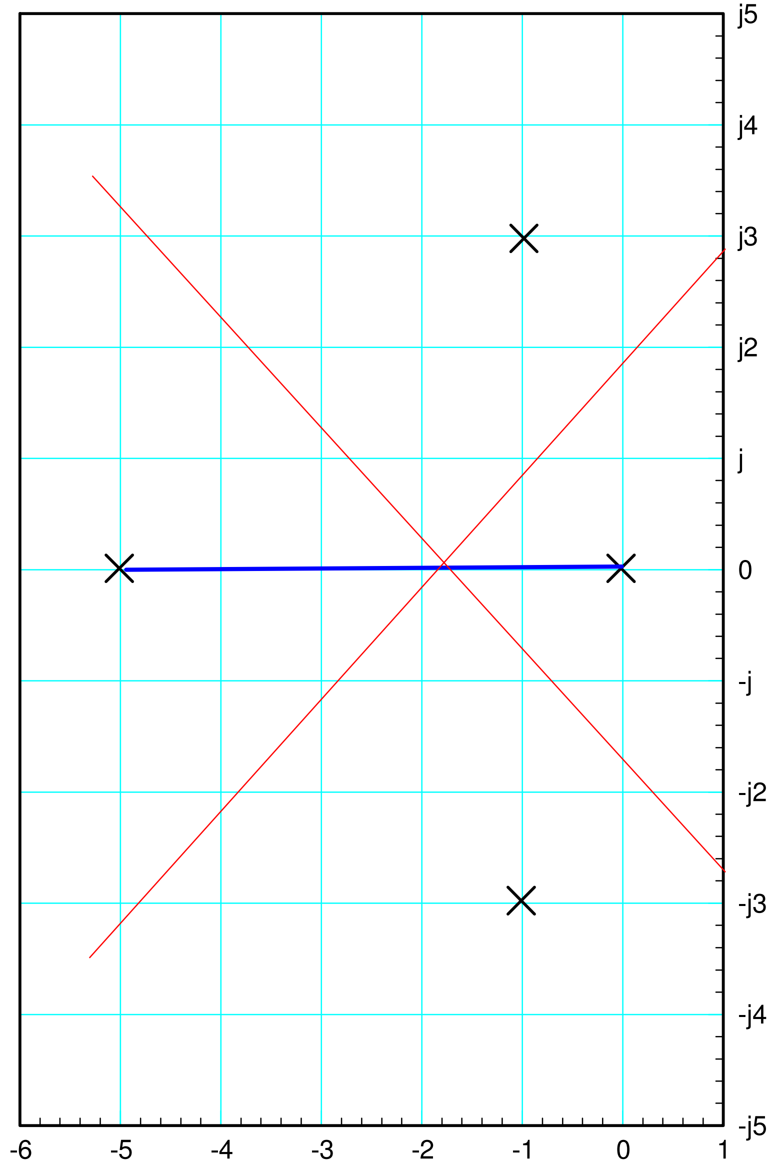
Example: $G(s) = \left(\frac{100}{s(s+1+j3)(s+1-j3)(s+5)}\right)$

Draw the root locus

- Real Axis Loci
- Asymptotes
- Departure Angle from the pole at s = -1 + j3



$$G(s) = \left(\frac{100}{s(s+1+j3)(s+1-j3)(s+5)} \right)$$



Real Axis Loci

- (0, -5)

Asymptptes

- 4 asymptotes
- +/- 45 degrees. +/- 135 degrees
- Intersect = -7/4

Departure Angles (take 1)

$$G(s) = \left(\frac{100}{s(s+1+j3)(s+1-j3)(s+5)} \right)$$

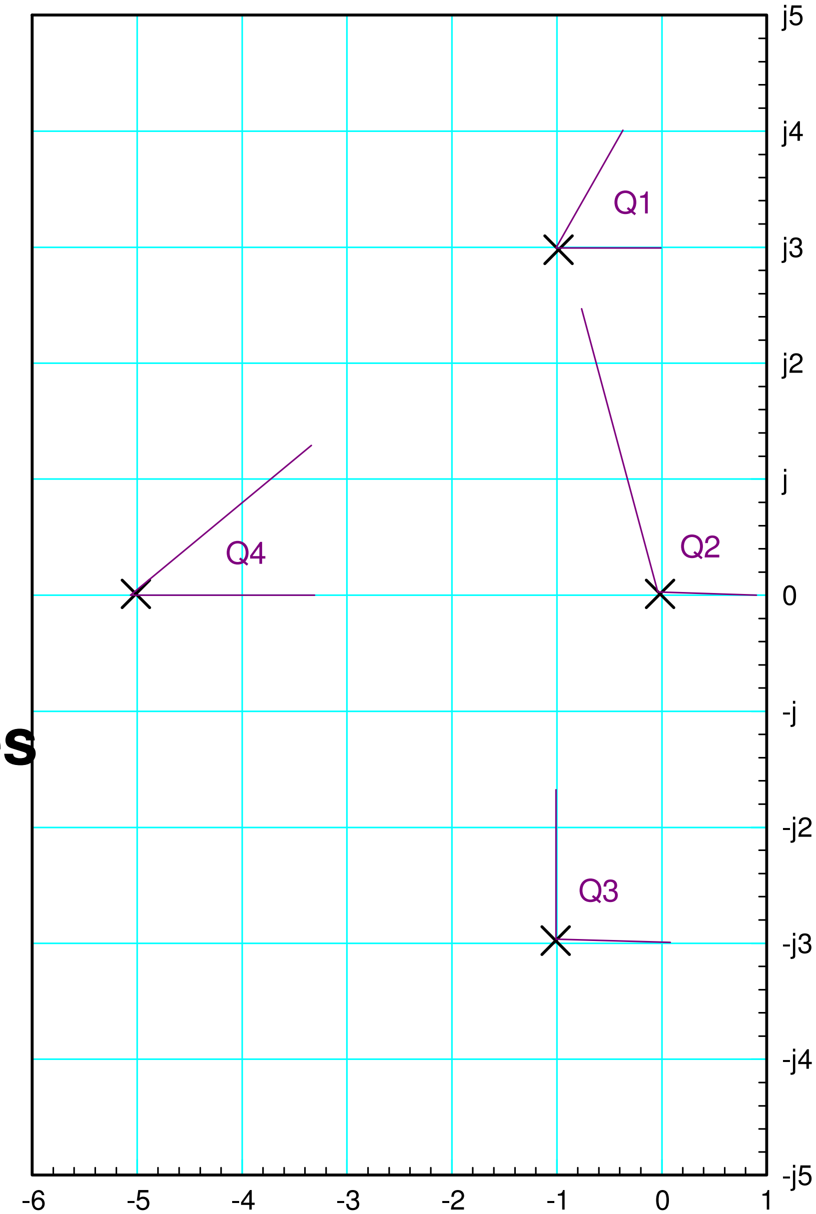
Angles must add up to 180 degrees

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 180^0$$

$$\theta_1 + 108.43^0 + 90^0 + 36.87^0 = 180^0$$

$$\theta_1 = -55.30^0$$

The departure angle us -55.30 degrees



Departure Angle (take 2)

$$G(s) = \left(\frac{100}{s(s+1+j3)(s+1-j3)(s+5)}\right)$$

At any point on the root locus, the angles add to 180 degrees

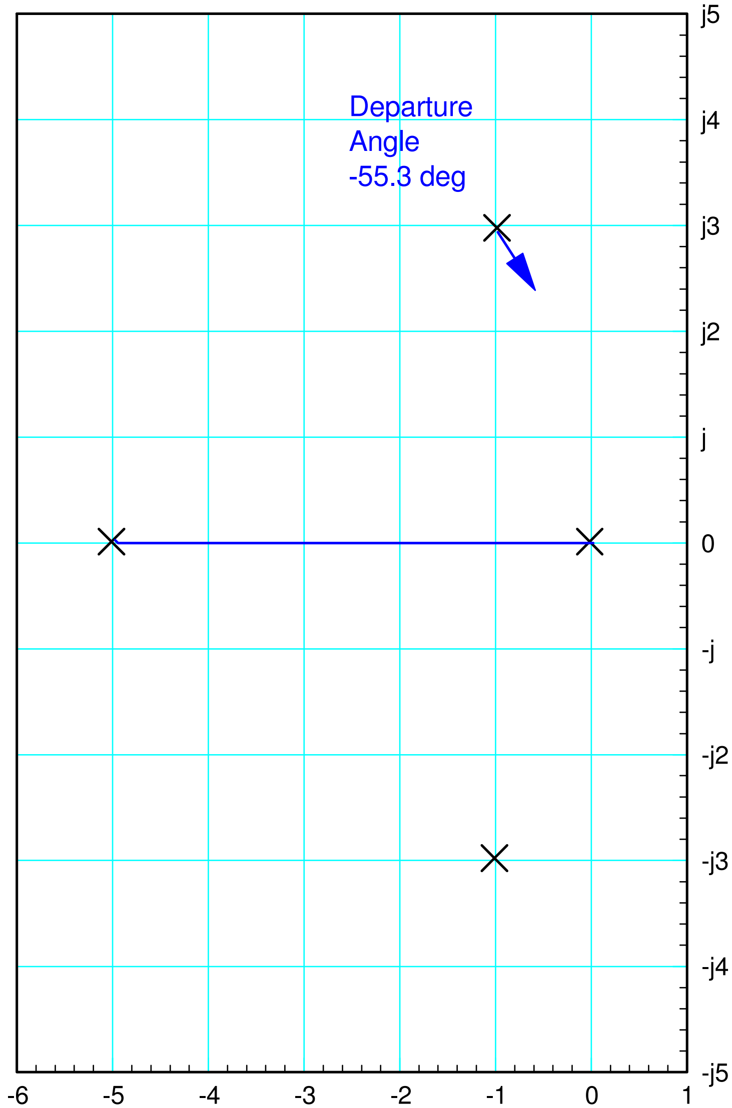
At s = -1 + j3

$$\angle\left(\frac{100}{s(s+1+j3)(s+1-j3)(s+5)}\right)_{s=-1+j3+\varepsilon} = 180^0$$

$$\angle\left(\frac{100}{s(s+1+j3)(s+5)}\right)_{s=-1+j3}\left(\frac{1}{s+1-j3}\right)_{s=-1+j3+\varepsilon} = 180^0$$

$$124.69^0 - \theta_1 = 180^0$$

$$\theta_1 = -55.30^0$$



Net Root Locus

Real Axis Loci

- (0, -5)

Asymptotes

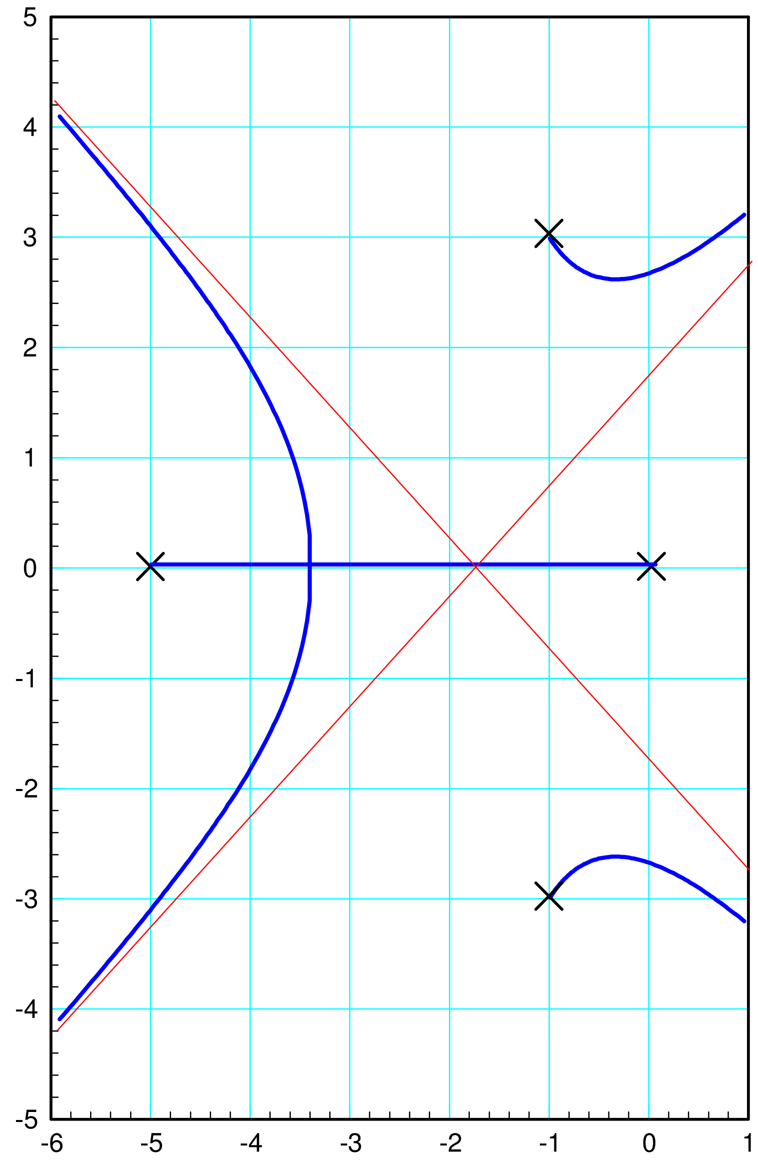
- 4 asymptotes
- +/- 45 degrees. +/- 135 degrees
- Intersect = -7/4

Breakaway Point

- Left of -2.5 (-3.4)

Departure Angle

- -55.30 degrees



Complex Zeros and Approach Angles

Note that

- Poles repel the root locus plot.
- Zeros attract the root locus plot.

If you have a zero, it will try to pull the root locus to the zero.

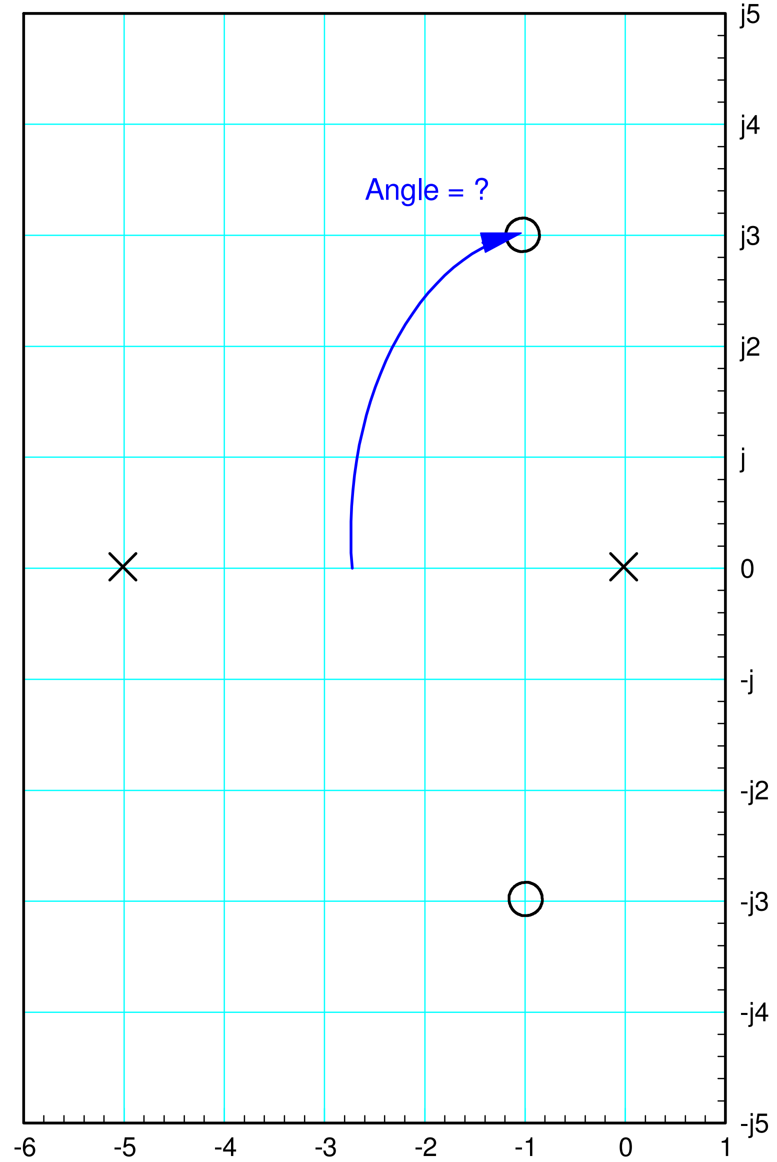
Answer: Whatever it takes for the angles to add up to 180 degrees

Approach Angles:

$$G(s) = \left(\frac{(s+1+j3)(s+1-j3)}{s(s+5)} \right)$$

Find the approach angle to the zero at

$s = -1 + j3$



Approach Angle (take 1)

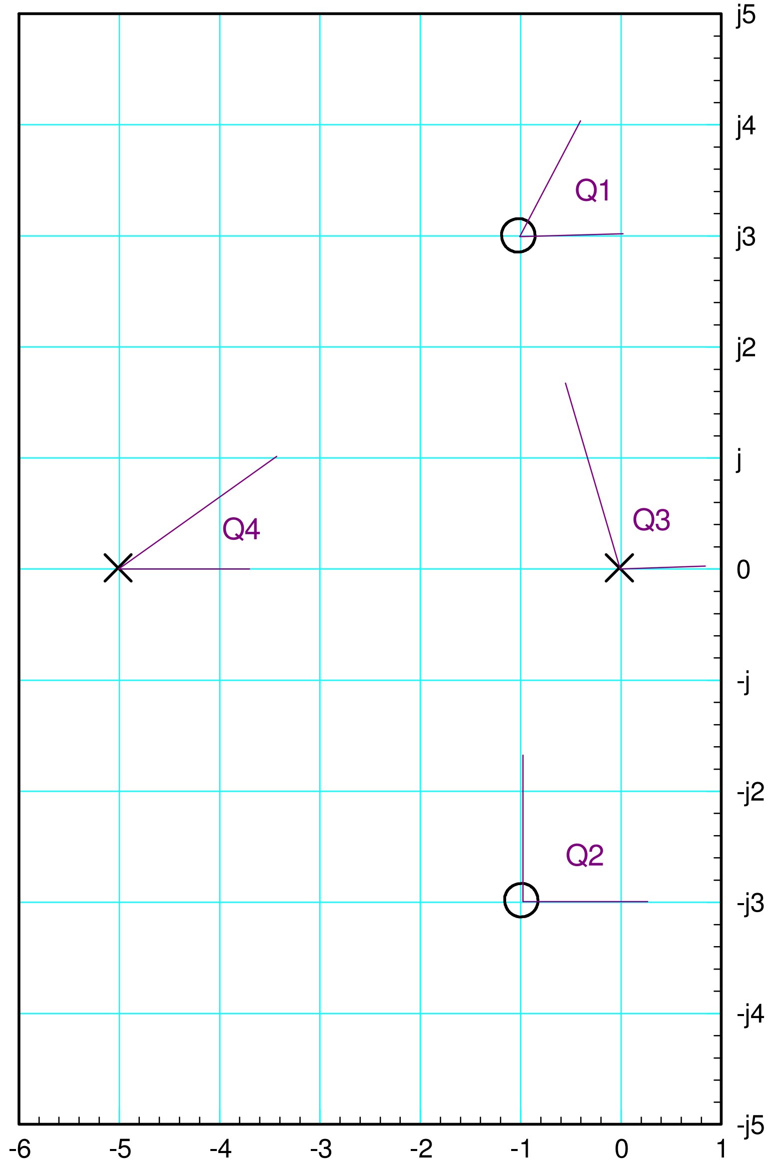
$$G(s) = \left(\frac{(s+1+j3)(s+1-j3)}{s(s+5)} \right)$$

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 180^0$$

$$\theta_1 + 90^0 - 108.43^0 - 36.87^0 = 180^0$$

$$\theta_1 = 235.30^0$$

$$= -124.70^0$$



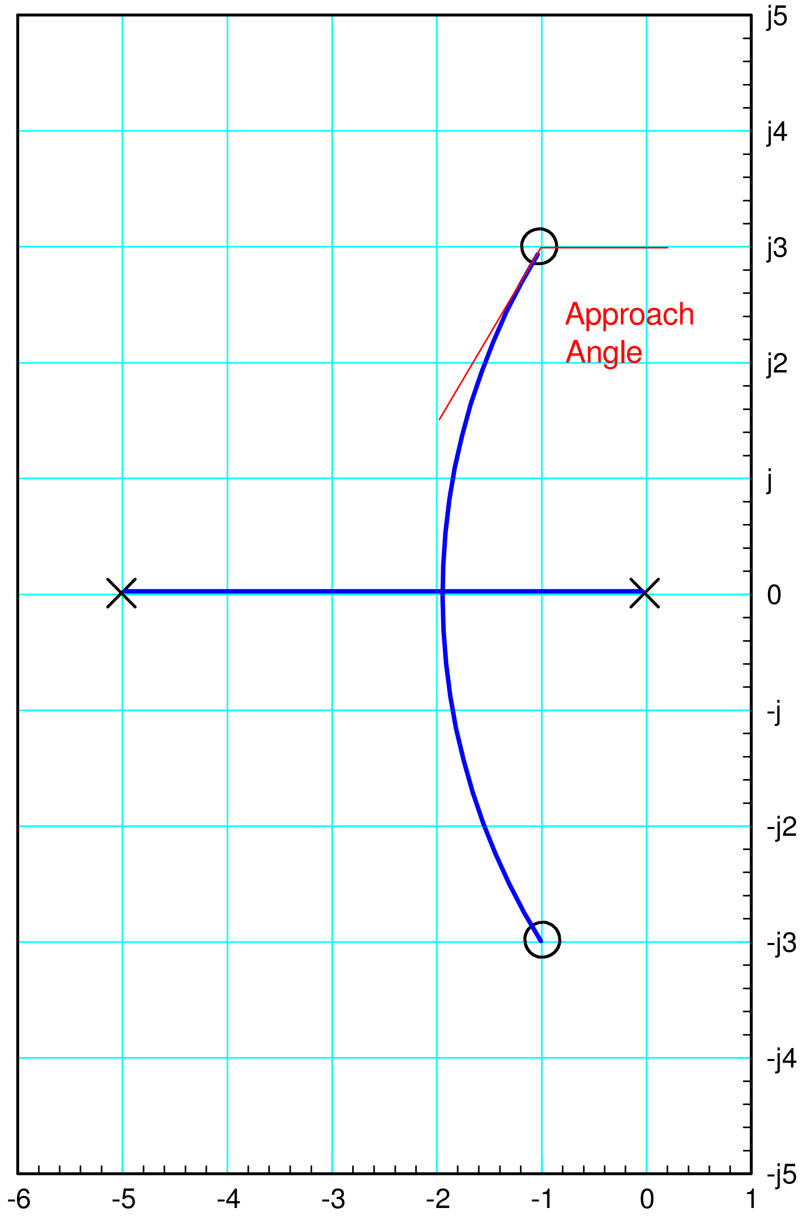
Approach Angle (take 2)

$$\angle\left(\frac{(s+1+j3)(s+1-j3)}{s(s+5)}\right)_{s=-1+j3+\varepsilon} = 180^0$$

$$\angle\left(\frac{(s+1+j3)}{s(s+5)}\right)_{s=-1+j3} + \angle\left(\frac{(s+1-j3)}{1}\right)_{s=-1+j3+\varepsilon} = 180^0$$

$$-55.30^0 + \theta_1 = 180^0$$

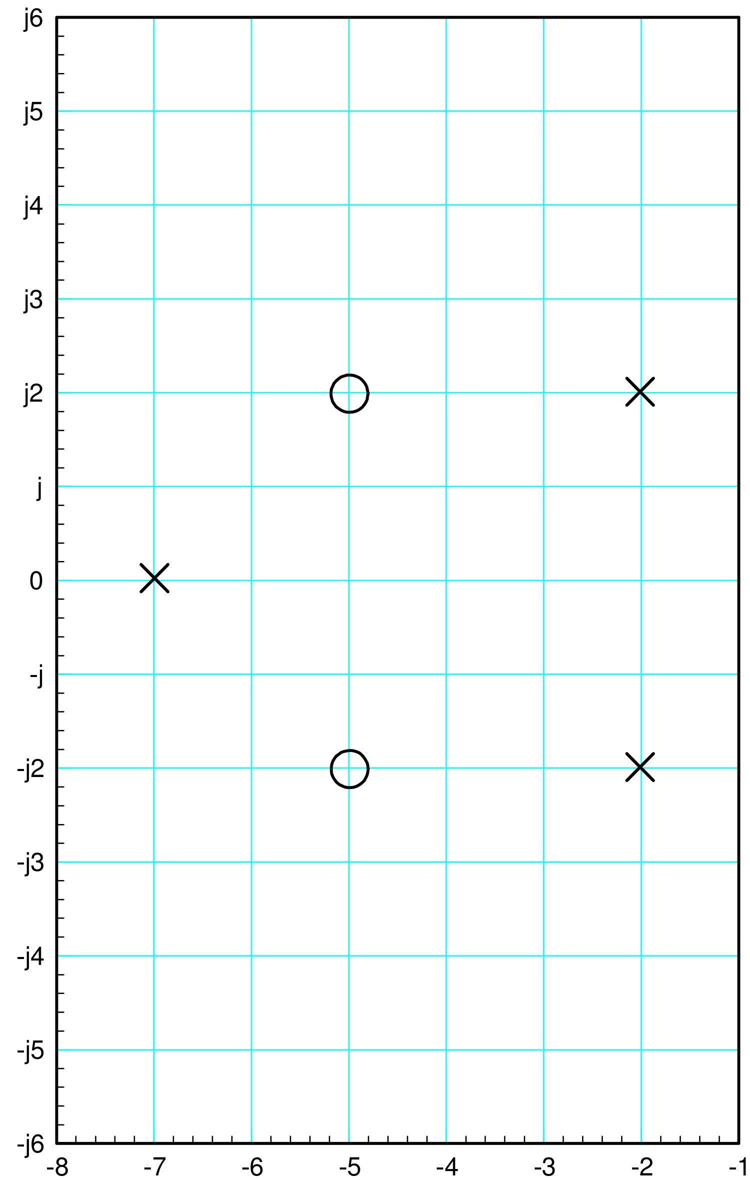
$$\theta_1 = 235.30^0$$
$$= -124.70^0$$



Example: Calculate the departure and approach angles

$$G(s) = \left(\frac{200\left(s^2+10s+29\right)}{(s+7)\left(s^2+4s+8\right)} \right)$$

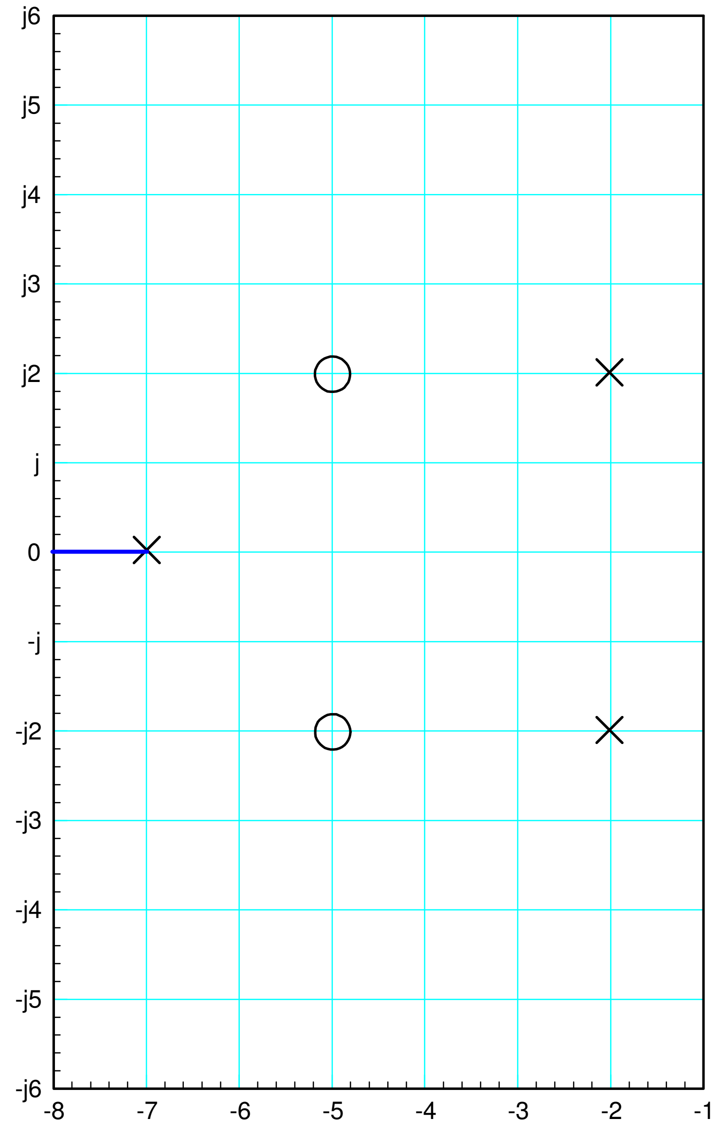
$$G(s) = \left(\frac{200(s+5+j2)(s+5-j2)}{(s+7)(s+2+j2)(s+2-j2)} \right)$$



Real Axis Loci

Odd number of poles & zeros to the right

(-7, -infinity)



Asymptotes

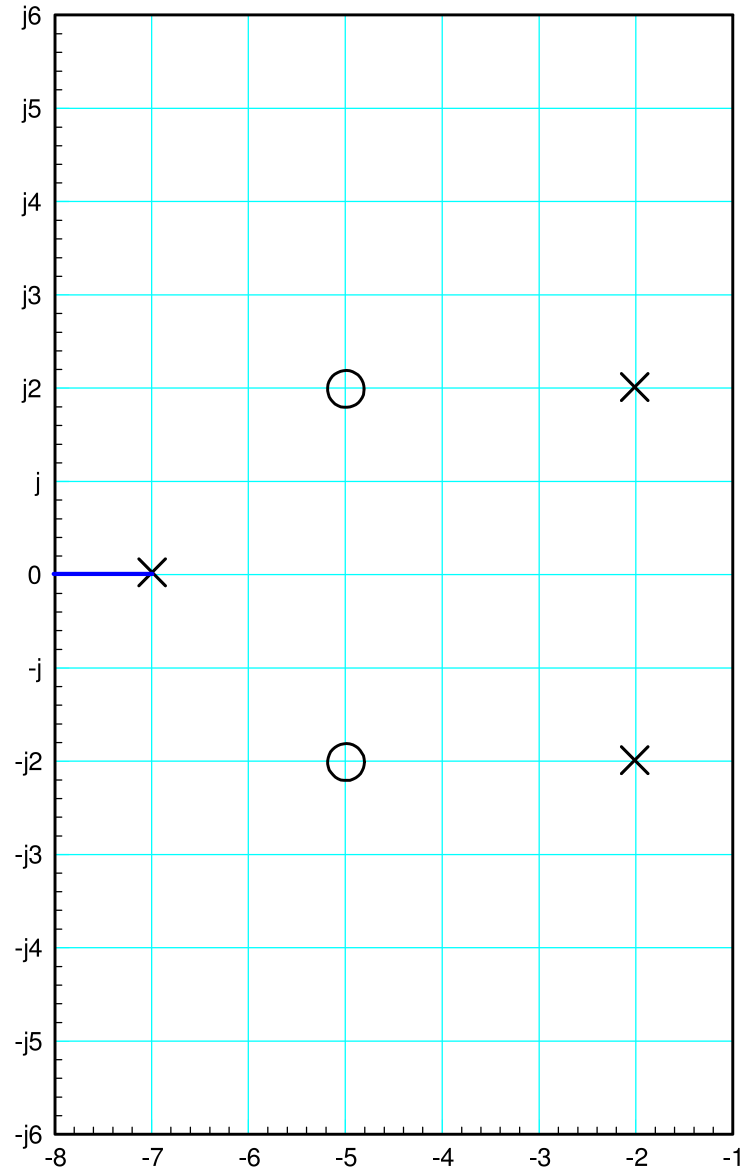
3 poles

2 zeros

= 1 extra pole

$$1 \cdot \theta = 180^0$$

$$\theta = 180^0$$



Departure Angle

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5 = 180^0$$

$$\angle\left(\frac{(s+5+j2)(s+5-j2)}{(s+7)(s+2+j2)(s+2-j2)}\right)_{s=-2+j2} = 180^0$$

$$\left(\frac{(s+5+j2)(s+5-j2)}{(s+7)(s+2+j2)}\right)_{s=-2+j2}\left(\frac{1}{s+2-j2}\right)_{s=-2+j2+\varepsilon} = 180^0$$

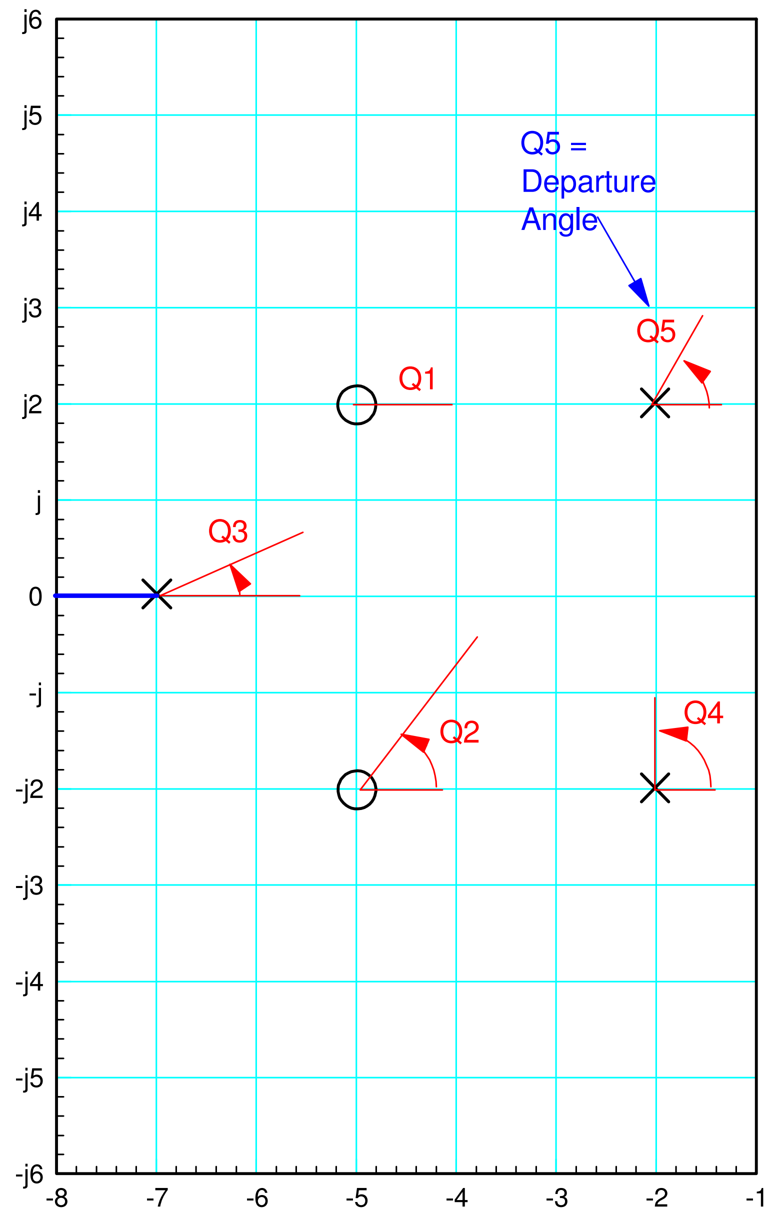
$$-58.6713^0 - \theta_5 = 180^0$$

$$\theta_5 = -238.67^0$$

or

$$\theta_5 = +121.33^0$$

The departure angle is 121.33 degrees



Approach Angle

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 - \theta_5 = 180^0$$

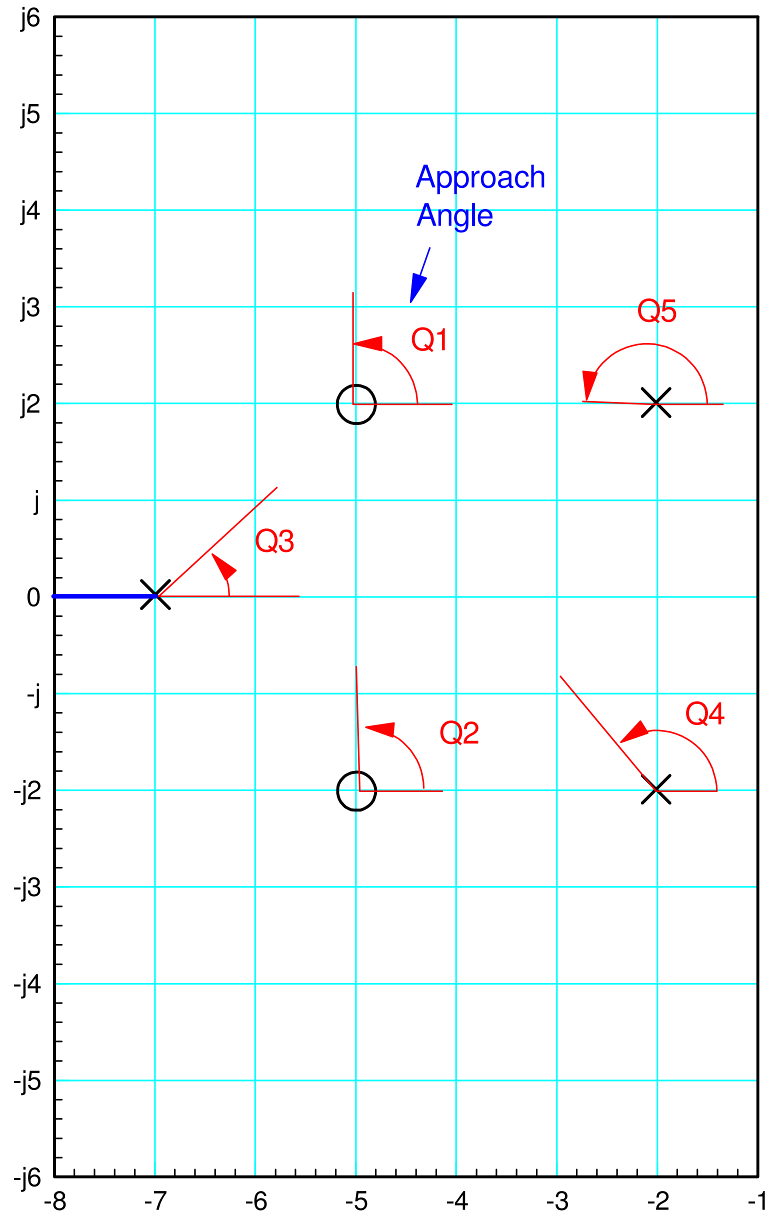
$$\angle\left(\frac{(s+5+j2)(s+5-j2)}{(s+7)(s+2+j2)(s+2-j2)}\right)_{s=-5+j2} = 180^0$$

$$\left(\frac{(s+5+j2)}{(s+7)(s+2+j2)(s+2-j2)}\right)_{s=-5+j2}\left(\frac{(s+5-j2)}{1}\right)_{s=-5+j2+\varepsilon} = 180^0$$

$$98.1301^0 + \theta_1 = 180^0$$

$$\theta_1 = 81.8699^0$$

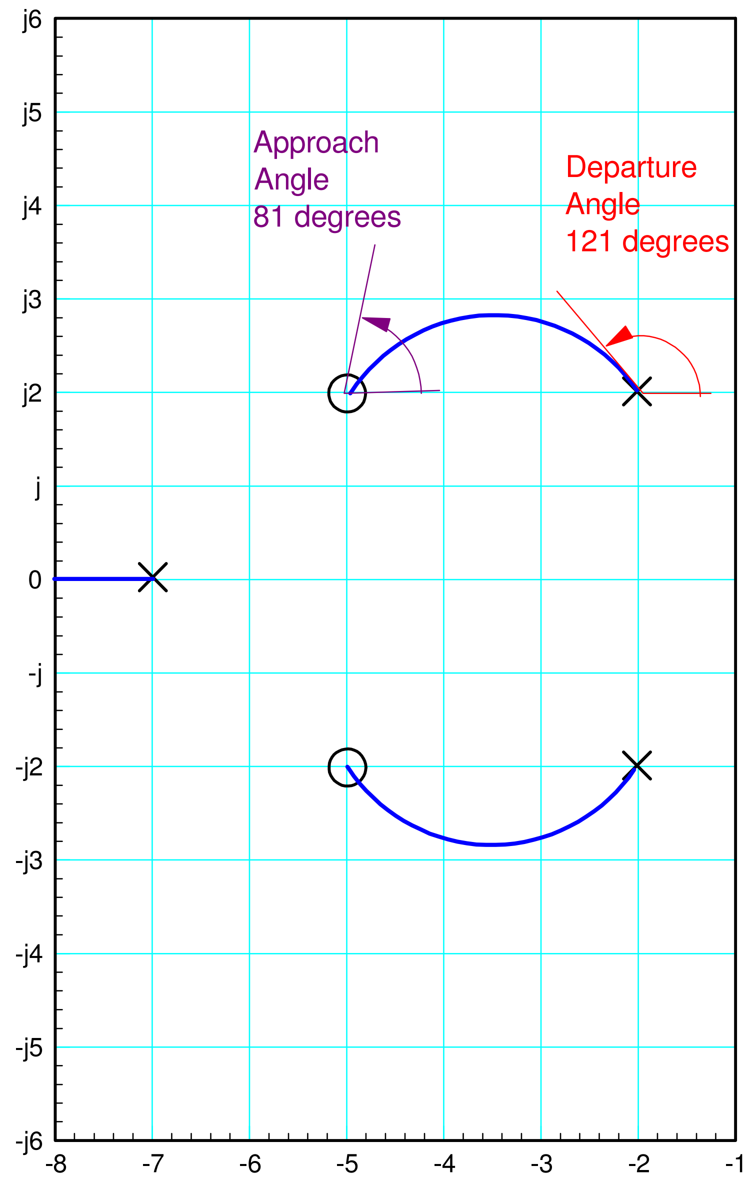
The approach angle is 81.8699 degrees



Resulting Root Locus Plot

Root locus leaves the complex pole at the departure angle

Root locus goes to the zero at the approach angle

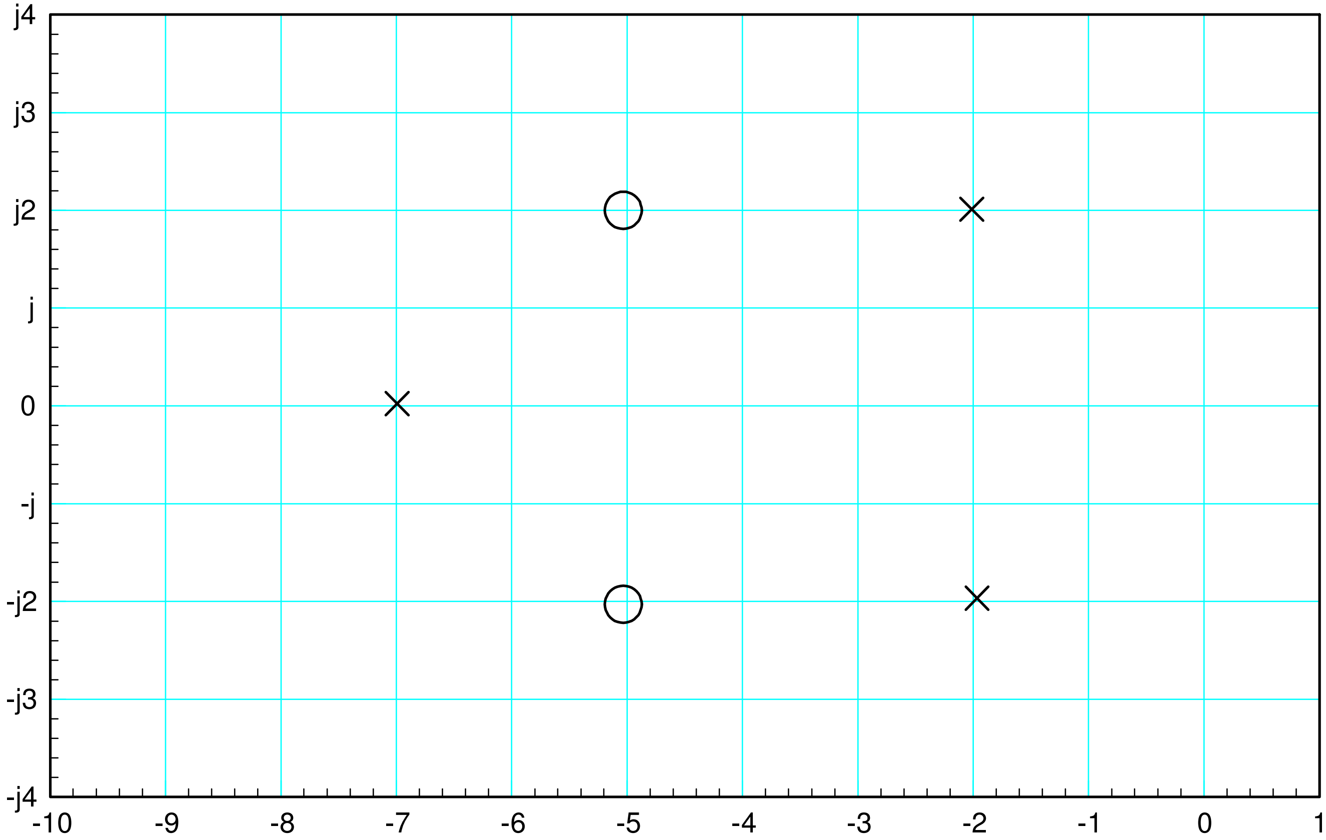


Handout: Sketch the root locus for $\left(\frac{200(s+5+j2)(s+5-j2)}{(s+7)(s+2+j2)(s+2-j2)}\right)$

Real Axis Loci:

Departure Angle from pole at -2 + j2

Approach Angle to zero at -5 + j2



Summary

Root locus plots follow the path of an electron

- Poles act like negative charges (repel)
- Zeros act like positive charges (attract)

At any point on the root locus, the angles *must* add to 180 degrees

- If you have complex poles the root locus leaves at the angle that keeps the sum equal to 180 degrees (the departure angle)

- If you have complex zeros the root locus approaches at the angle that keeps the sum equal to 180 degrees (the approach angle)

Knowing these angles can help in sketching the root locus plot.