Routh Criteria & Stability

ECE 461/661 Controls Systems

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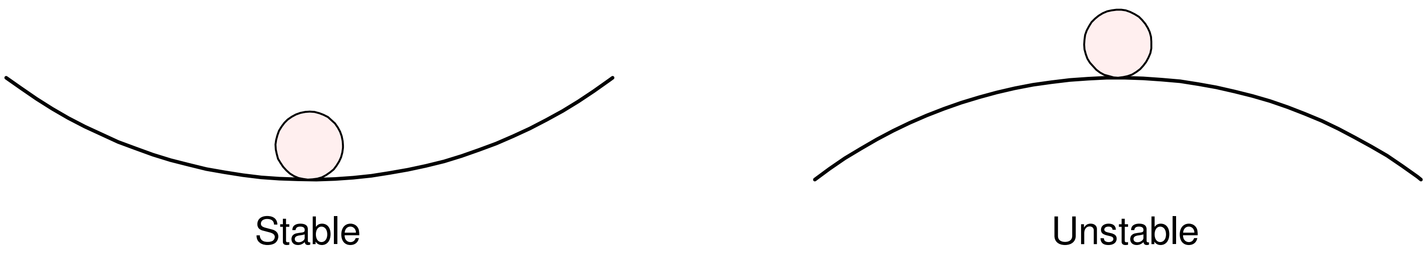
Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Stability

- If you perturb a system, it returns to its original position
- Unstable system are pretty much useless: they break



Stability: If perturbed, it will return to its original position

Feedback

You *can* stabilize unstable systems using feedback

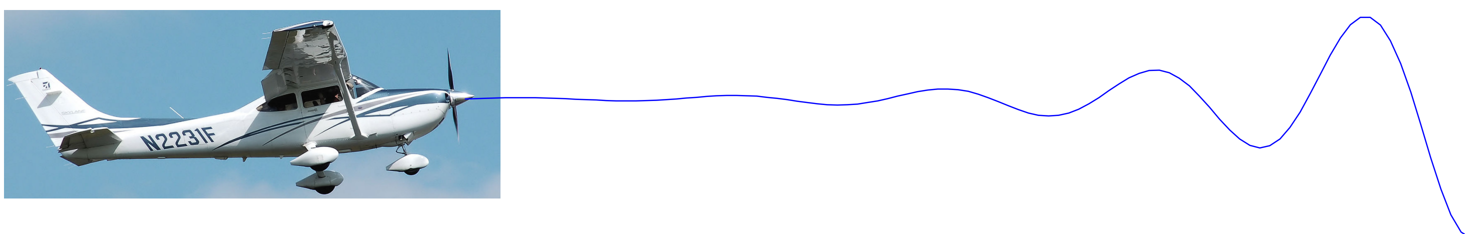
Examples of unstable systems:

- Standing: if you pass out, you fall down.
- Bicycles: takes time (and skinned knees) to master
- Rockets: balance on a ball of flame
- Economies:
  - Good times: People buy more, companies sell more and hire more
  - Bad times: Peoble buy less, companies lay off employees, people buy even less
  - Boom / bust cycles date back to the Roman Empire
  - The Federal Reserve's job is to adjust the money supply and interest rates to keep growth at a steady, sustainable level

Problem with Feedback

Too much feedback can turn a stable system into an ustable one

- Driving a car: when learning to drive you...
  - Jerk the steering wheel, sending the car into the curb, then
  - You jerk the steering wheel, sending the car into the other curb
- Pilot-Induced Oscillations
  - A Cessna is open-loop stable: if you take your hands off the controls it flies itself.
  - When landing, sometimes a pilot will over-correct, sending the plane towards the grond
  - Then over-correct, sending the plane into the sky
  - Each oscillation gets worse and worse
  - To fix, take your hands off the controls and let the plane fly itself.

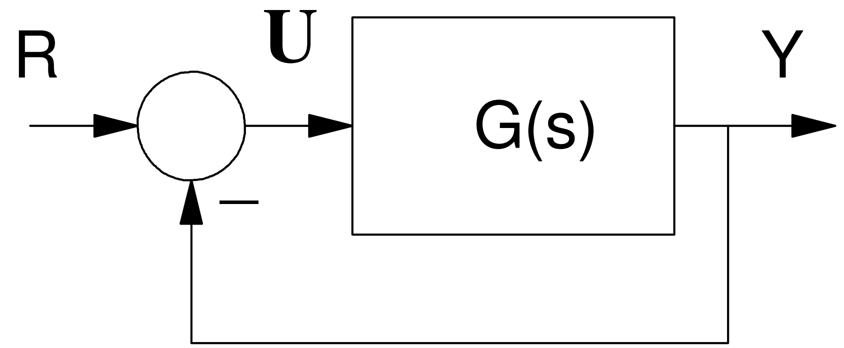


Example: Too Much Feedback

G(s) is open-loop stable

- Heat equation
- 3-stage RC filter

$$Y = \left(\frac{100}{(s+1)(s+2)(s+3)}\right)U$$



Closed-loop system is unstable:

$$Y = \left(\frac{G}{1+G}\right)R = \left(\frac{100}{(s+1)(s+2)(s+3)+100}\right)R$$

$$Y = \left(\frac{100}{(s-0.3567+j3.9575)(s-0.3567-j3.9575)(s+6.7134)}\right)R$$

Problem:

How do you determine whether a system is stable?

- Matlab: All roots must be negative definite
- Routh Table: Also works

How do you determine the range of feedback gains that result in a stable system?

- Routh Table

The latter question is harder but useful:

- Feedback control systems often have knobs you can adjust. It would be nice if you could dummy-proof such a system and make sure the operator doesn't make the system go unsable.
- Sometimes, a system just isn't stabizable. It would be nice to know this before spending hours trying to stabilize it.

Routh Table

Determine if a given polynomial is negative definite (i.e. stable)

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 = 0$$

Row 1 & 2: Every other term starting with a_n

a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...

Row 3+: Generate from prior two rows

- Repeat until all entries are zero

Previous two rows:	a	b	c	d	...
	e	f	g	h	...
New Row	$\frac{-\begin{vmatrix} a & b \\ e & f \end{vmatrix}}{e}$	$\frac{-\begin{vmatrix} a & c \\ e & g \end{vmatrix}}{e}$	$\frac{-\begin{vmatrix} a & d \\ e & h \end{vmatrix}}{e}$	etc.	

Routh Criteria:

Number of unstable poles = Number of sign flips in column #1

- For a system to be stable, there can be no sign flips in column #1
- The range of gains that produce a stable system are the range that results in no sign flips in column #1

Example: Determine if this polynomial is stable using a Routh table

$$(s+1)(s+2)(s+3)(s+4) = s^4 + 10s^3 + 35s^2 + 50s + 24$$

1	35	24	0
10	50	0	0
$\frac{-\begin{vmatrix} 1 & 35 \\ 10 & 50 \end{vmatrix}}{10} = 30$	$\frac{-\begin{vmatrix} 1 & 24 \\ 10 & 0 \end{vmatrix}}{10} = 24$	$\frac{-\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0$	0
$\frac{-\begin{vmatrix} 10 & 50 \\ 30 & 24 \end{vmatrix}}{30} = 42$	$\frac{-\begin{vmatrix} 10 & 0 \\ 30 & 0 \end{vmatrix}}{30} = 0$	$\frac{-\begin{vmatrix} 10 & 0 \\ 30 & 0 \end{vmatrix}}{30} = 0$	
$\frac{-\begin{vmatrix} 30 & 24 \\ 42 & 0 \end{vmatrix}}{42} = 24$	$\frac{-\begin{vmatrix} 30 & 0 \\ 42 & 0 \end{vmatrix}}{42} = 0$		
$\frac{-\begin{vmatrix} 42 & 0 \\ 24 & 0 \end{vmatrix}}{24} = 0$	$\frac{-\begin{vmatrix} 42 & 0 \\ 24 & 0 \end{vmatrix}}{24} = 0$		

There are no sign flips. Hence, this polynomial is stable. (The system is stable).

Example 2: Determine the range of K for stability: $Y = \left(\frac{K}{(s+1)(s+2)(s+3)+K} \right) R$

- Multply out the denominator

$$(s+1)(s+2)(s+3)+K = s^3 + 6s^2 + 11s + 6 + K$$

- Form a Routh Table *you can scale a row by a positive constant*
- No sign flips in column #1
- Result: -6 < K < 60

1	11	0	
6	6+K	0	
60-K	0	0	K < 60
6+K	0	0	K > -6
0	0	0	

Note: If you plug in the endpoints, you get roots on the jw axis

K = -6:

- roots = {0, -3+j1.4142, -3-j1.4142}

K = +60:

- roots = {-6, +j3.3166, -j3.3166}

Example 3: $\left(\frac{GK}{1+GK}\right) = \left(\frac{Ds^2+Ps+I}{s(s+1)(s+2)(s+3)+\left(Ds^2+Ps+I\right)}\right)$

Multiply out the denominator:

$$s(s+1)(s+2)(s+3)+(Ds^2+Ps+I)=0$$

Build a Routh Table

1	11+D	I	Routh Criteria
6	6+P	0	
$10+D-\frac{P}{6}$	I	0	$10+D-\frac{P}{6}>0$
$\frac{60+6D+P+10P+DP+\frac{P^2}{6}-6I}{10+D+\frac{P}{6}}$	0	0	$60+6D+P+10P+DP+\frac{P^2}{6}-6I>0$
I	0	0	$I>0$

Handout: Fill in a Routh table for the following polynomial

- Determine the range of k for stability

$$s^4 + 14s^3 + 59s^2 + 46s + 2k - 120 = 0$$

Sidelight: When is a mass-spring system stable?

$$ms^2 + bs + k = 0$$

m	k		
b	0		
k	0		
0	0		

There are no sign flips if {m, b. k} are

- all positive, or
- all negative

A world with negative mass, friction, and springs would also work.

Summary

Routh criteria is one way of telling if a polynomial is negative definite

- i.e. the system is stable

It's useful for finding the range of gains that result in a stable system

- dummy protection
- don't let the operator go outside this range

It just stay *stable* or *unstable*

- *We need a different tool if you care about a 'good' response*