Rotational Systems & Gears

ECE 461/661 Controls Systems

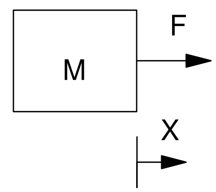
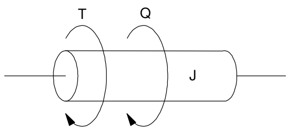
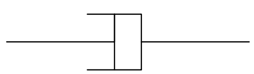
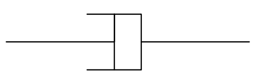
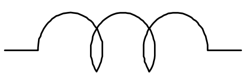
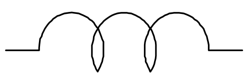
Jake Glower - Lecture #16

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

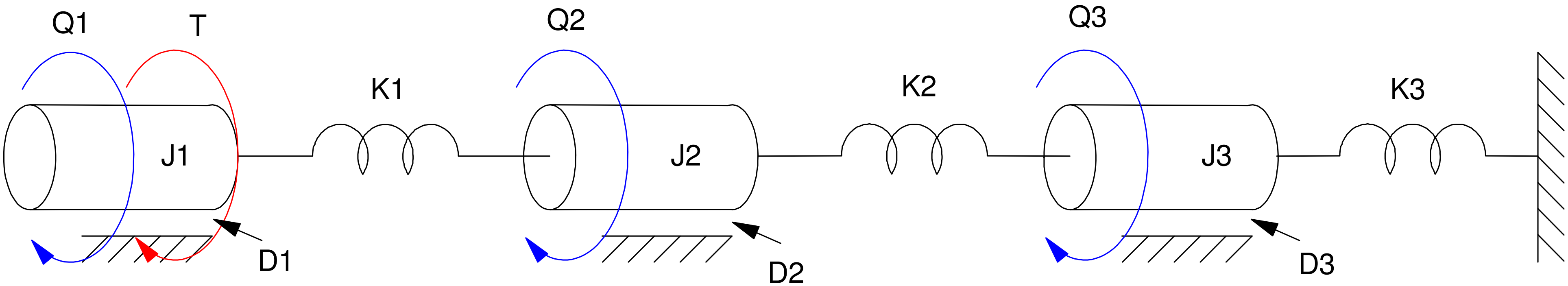
Rotational Systems

Rotational systems behave exactly like translational systems, except that

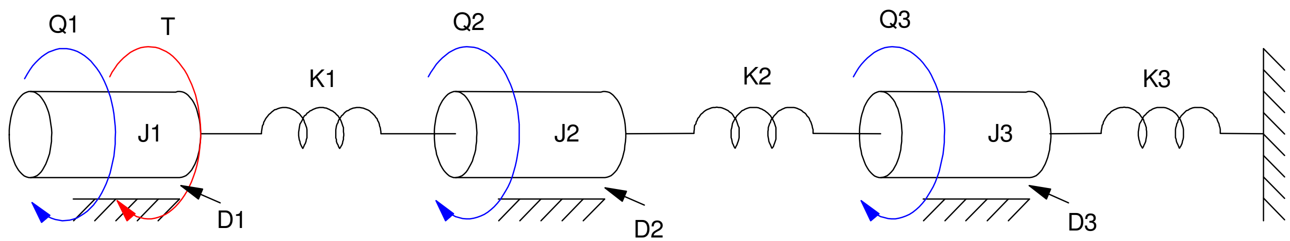
- Force becomes Torque
- Position becomes Angle

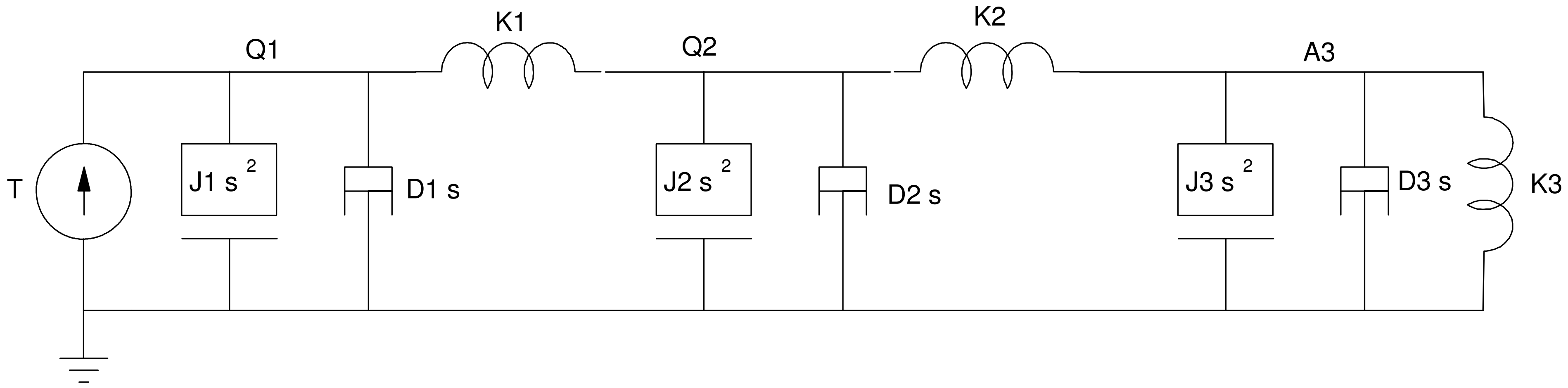
	Translational Systems		Rotational System	
Inertia	 M, F, X	$F = Ms^2X$	 T, Q, J	$T = Js^2\theta$
Friction	B 	$F = BsX$	D 	$T = Ds\theta$
Spring	K 	$F = KX$	K 	$T = K\theta$

Example: Draw the circuit equivalent



Inertia - Spring Rotational System





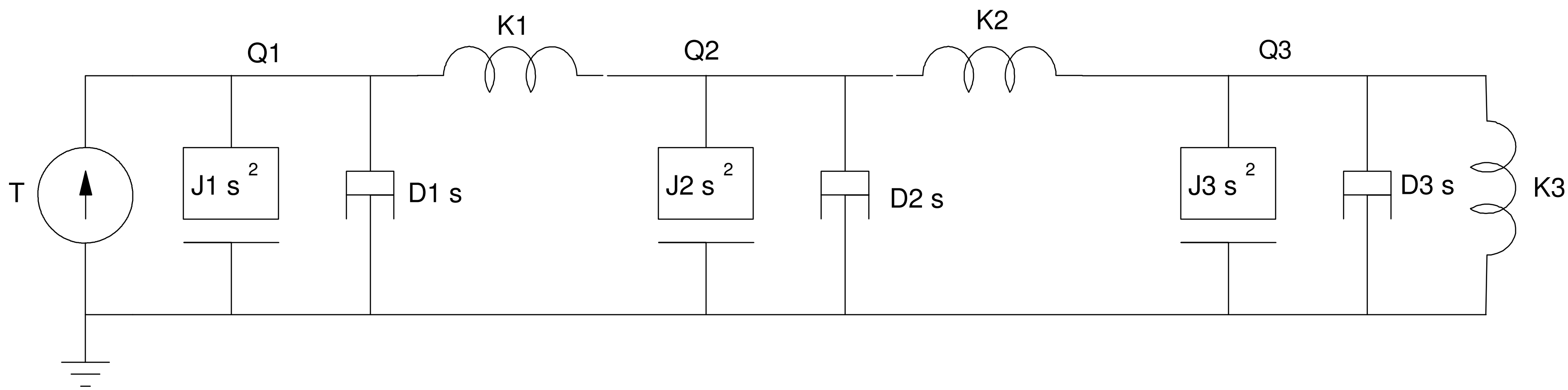
Circuit Equivalent

Node Equations:

$$(J_1 s^2 + D_1 s + K_1)\theta_1 - (K_1)\theta_2 = T$$

$$(J_2 s^2 + D_2 s + K_1 + K_2)\theta_2 - (K_1)\theta_1 - (K_2)\theta_3 = 0$$

$$(J_3 s^2 + D_3 s + K_2 + K_3)\theta_3 - (K_2)\theta_2 = 0$$



Solve for the highest derivative:

$$s^2\theta_1 = -\left(\frac{D_1}{J_1}s + \frac{K_1}{J_1}\right)\theta_1 + \left(\frac{K_1}{J_1}\right)\theta_2 + \left(\frac{1}{J_1}\right)T$$

$$s^2\theta_2 = -\left(\frac{D_2 s}{J_2} + \frac{K_1+K_2}{J_2}\right)\theta_2 + \left(\frac{K_1}{J_2}\right)\theta_1 + \left(\frac{K_2}{J_2}\right)\theta_3$$

$$s^2\theta_3 = -\left(\frac{D_3 s}{J_3} + \frac{K_2+K_3}{J_3}\right)\theta_3 + \left(\frac{K_2}{J_3}\right)\theta_2$$

Place in matrix form

$$\begin{bmatrix} s\theta_1 \\ s\theta_2 \\ s\theta_3 \\ \cdots \\ s^2\theta_1 \\ s^2\theta_2 \\ s^2\theta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ -\left(\frac{K_1}{J_1}\right) & \left(\frac{K_1}{J_1}\right) & 0 & \vdots & -\left(\frac{D_1}{J_1}\right) & 0 & 0 \\ \left(\frac{K_1}{J_2}\right) & -\left(\frac{K_1+K_2}{J_2}\right) & \left(\frac{K_2}{J_2}\right) & \vdots & 0 & -\left(\frac{D_2}{J_2}\right) & 0 \\ 0 & \left(\frac{K_2}{J_3}\right) & -\left(\frac{K_2+K_3}{J_3}\right) & \vdots & 0 & 0 & -\left(\frac{D_3}{J_3}\right) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \cdots \\ s\theta_1 \\ s\theta_2 \\ s\theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ \left(\frac{1}{J_1}\right) \\ 0 \\ 0 \end{bmatrix} T$$

Gears

Gears are like transformers:

- They convert one speed to another as the gear ratio,
- They convert one impedance to another as the gear ratio squared.

Requirements

- Distance traveled at the interface must match
- Power In = Power Out

Distance Traveled at the Interface must Match

$$\text{distance} = r\theta$$

$$d_1 = r_1\theta_1$$

$$d_2 = r_2\theta_2$$

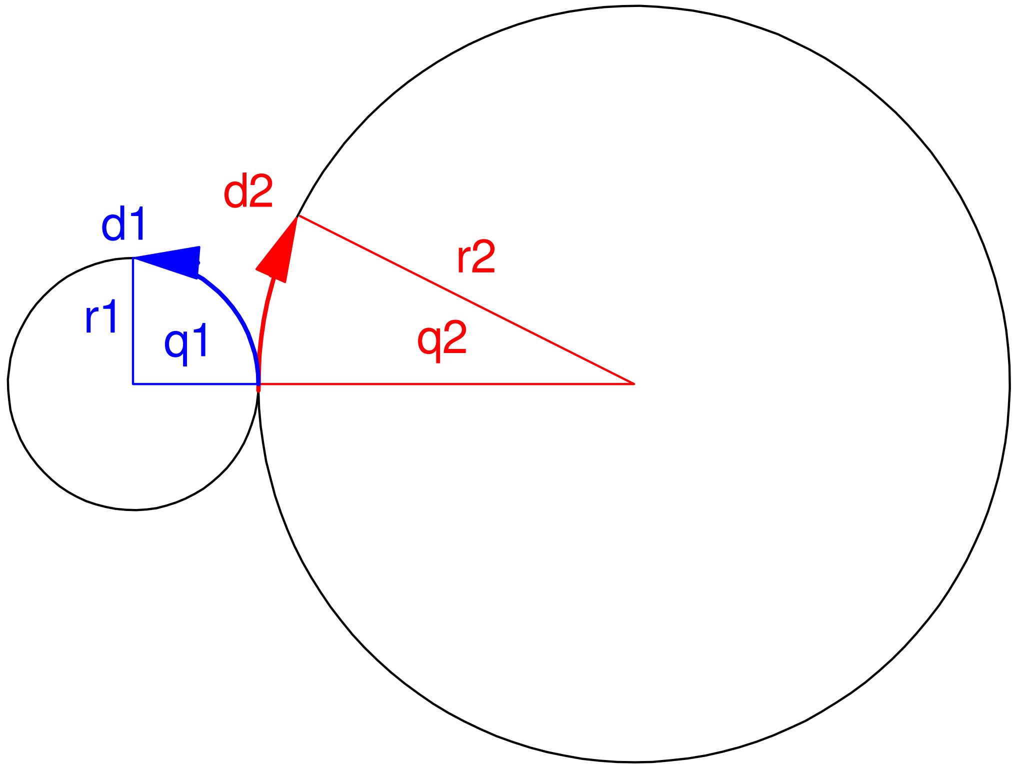
$$d_1 = d_2$$

$$r_1\theta_1 = r_2\theta_2$$

or

$$\theta_1 = \left(\frac{r_2}{r_1}\right)\theta_2$$

$$\dot{\theta}_1 = \left(\frac{r_2}{r_1}\right)\dot{\theta}_2$$



The forces at the interface must match

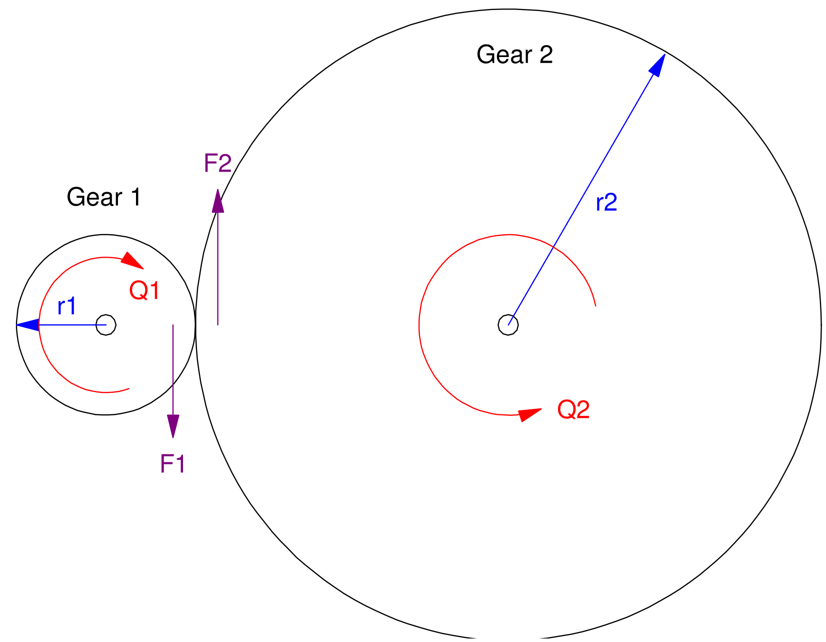
$$T_1 = F_1 r_1$$

$$T_2 = F_2 r_2$$

$$F_1 = F_2$$

$$\frac{T_1}{r_1} = \frac{T_2}{r_2}$$

$$T_1 = \left(\frac{r_1}{r_2}\right) T_2$$



Energy must balance

$$T_1 \dot{\theta}_1 = T_2 \dot{\theta}_2$$

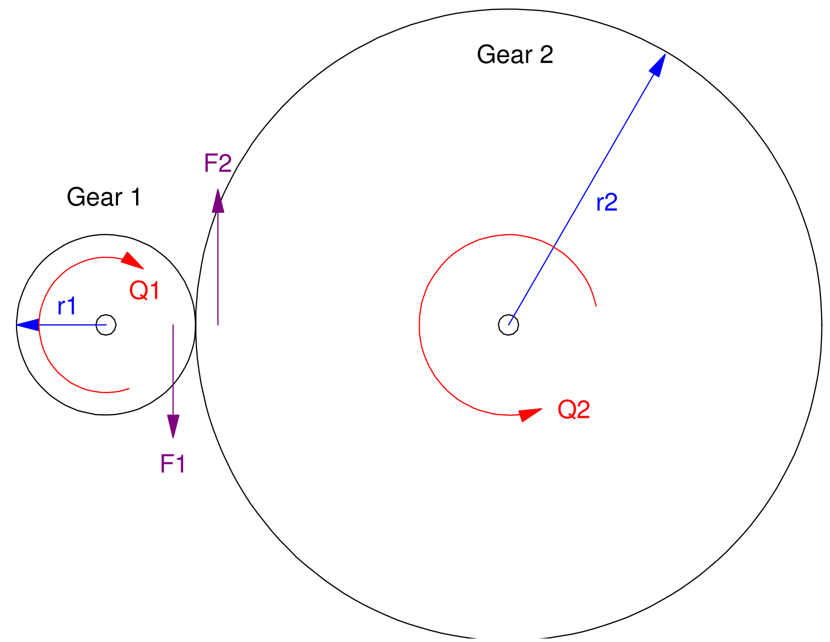
Impedance & Gears

Assume Gear #1 has inertia J1. What does gear #2 see?

$$T_1 = J_1 s^2 \theta_1$$

$$\left(\frac{r_1}{r_2}\right) T_2 = (J_1 s^2)\left(\frac{r_2}{r_1}\right)\theta_2$$

$$T_2 = \left(\frac{r_2}{r_1}\right)^2 (J_1 s^2)\theta_2$$



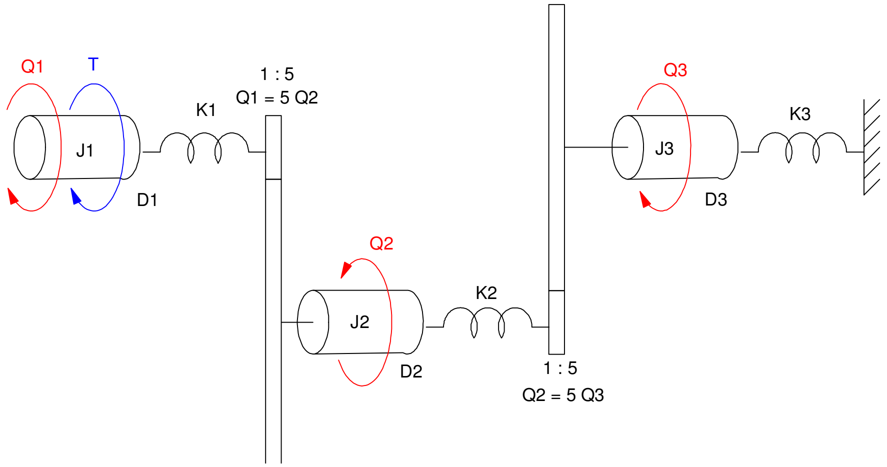
Relative to gear #2, inertial J1 looks like an inertia of

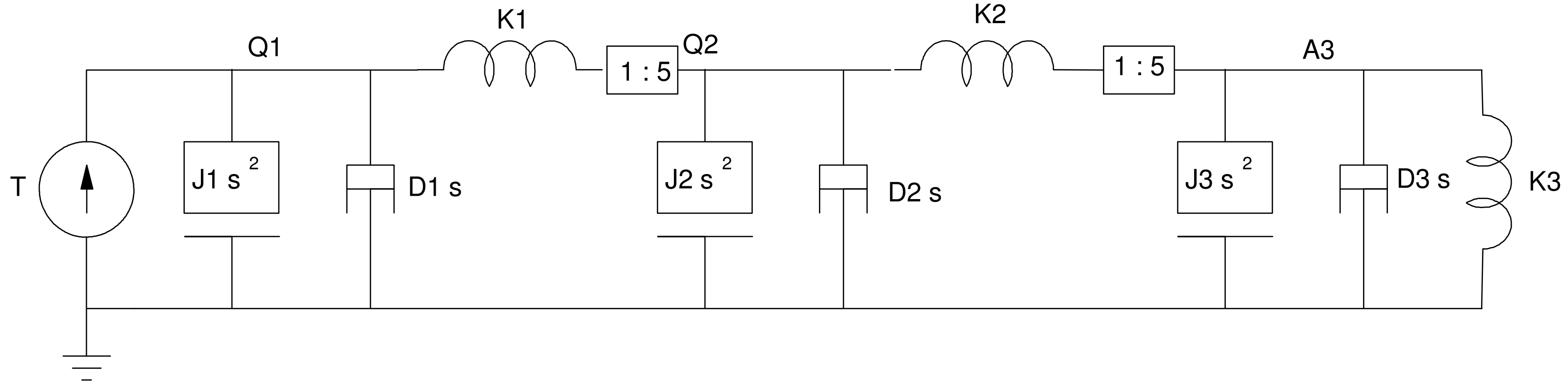
$$J = \left(\frac{r_2}{r_1}\right)^2 J_1$$

Impedances go through a gear as the turn ratio squared

Rotational System with Gears

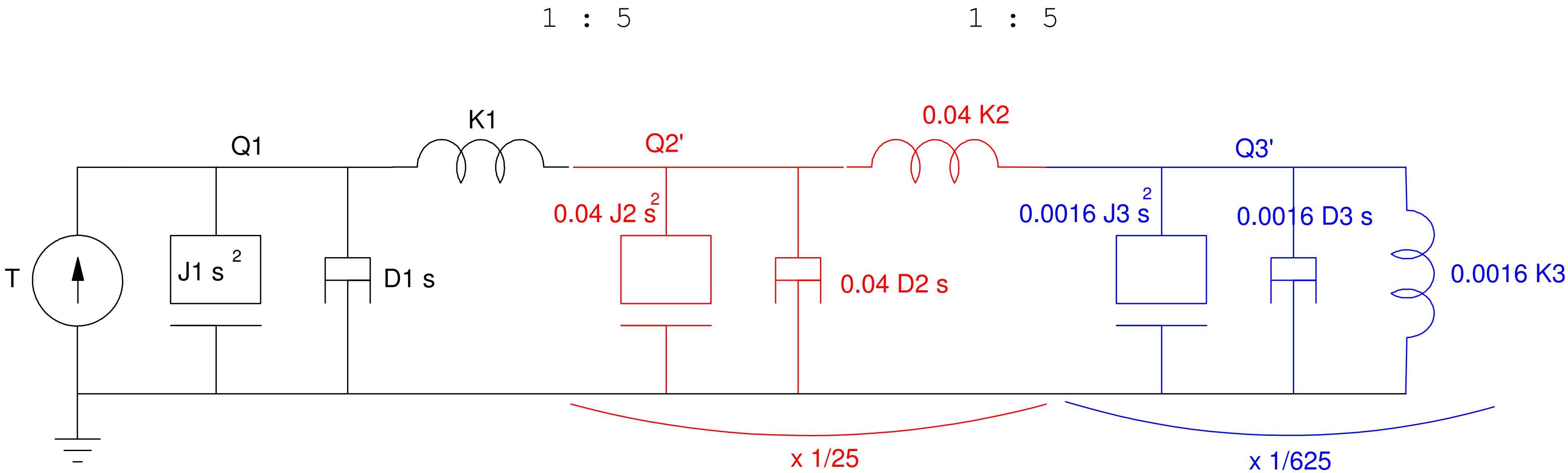
i) Draw the circuit equivalent with the gears





Remove the gears

$$Y_{new} \Rightarrow \left(\frac{\text{where going to}}{\text{where coming from}}\right)^2 Y_{old}$$



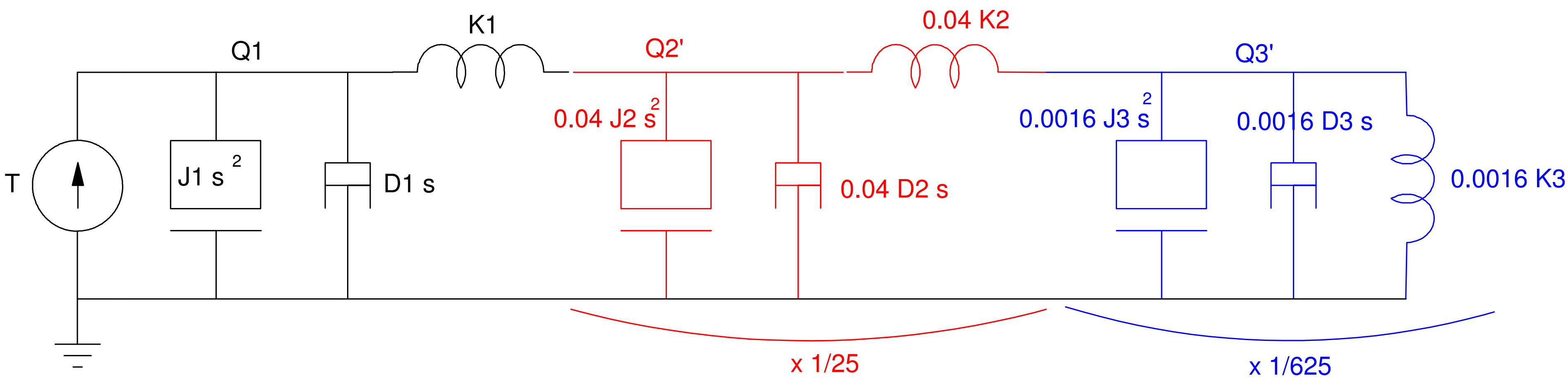
Now write the node equations

- note: angles are now different: scalar multiples of q1, q2, q3
- from here on, it's the same as before

$$(J_1 s^2 + D_1 s + K_1)\phi_1 - (K_1)\phi_2 = T$$

$$\left(\frac{J_2}{5^2}s^2 + \frac{D_2}{5^2}s + K_1 + \frac{K_2}{5^2}\right)\phi_2 - (K_1)\phi_1 - \left(\frac{K_2}{5^2}\right)\phi_3 = 0$$

$$\left(\frac{J_3}{25^2}s^2 + \frac{D_3}{25^2}s + \frac{K_3}{25^2} + \frac{K_2}{5^2}\right)\phi_3 - \left(\frac{K_2}{5^2}\right)\phi_2 = 0$$

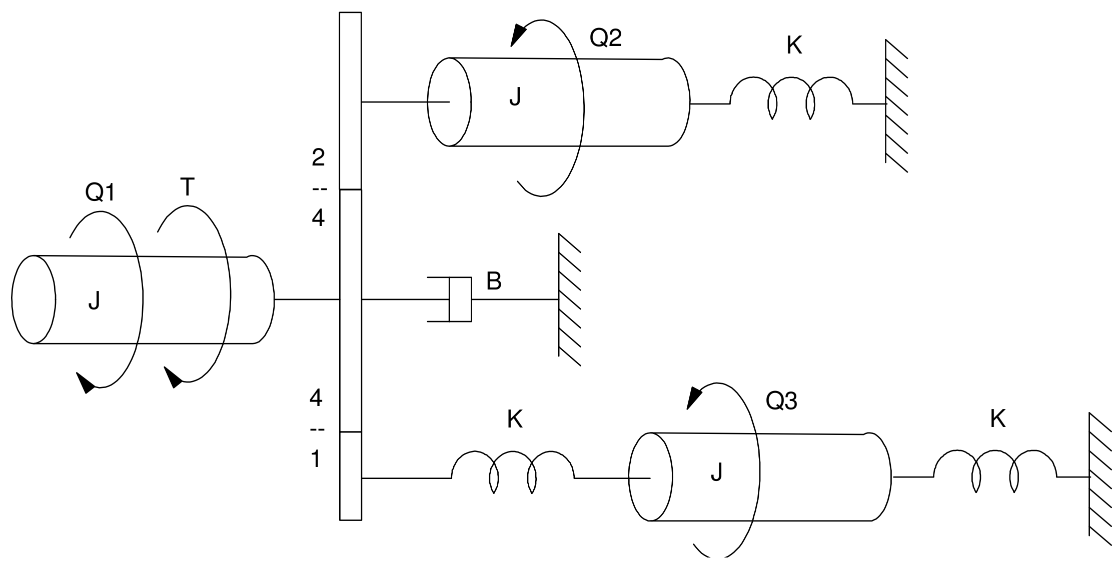


Handout:

Draw the circuit equivalent for the following mass-spring system. Assume

- J = 1kg, B = 0.2 Nms/rad, K = 10N/rad

Write the equations of motion (i.e. write the voltage node equations)



DC Servo Motors

- Type of motor
- Used to control rotational systems

Advantages:

- Current is Torque
- Voltage is speed (sort of)
- Easy to model
- Easy to power (DC signal)

Disadvantatges

- Larger than AC motors
- More expensive than AC motors
- Less efficient than AC motors

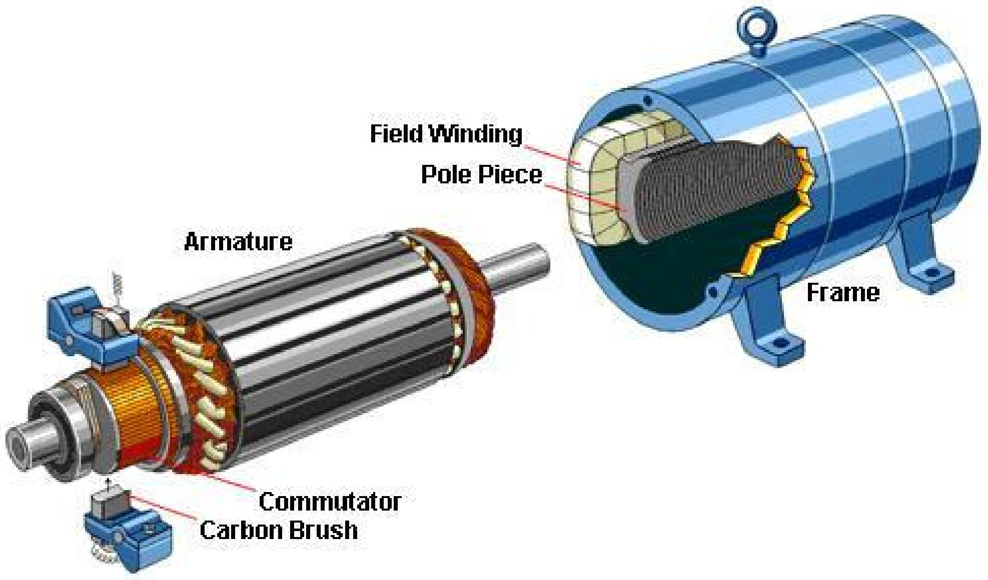
DC Servo Motor	AC Motor
43 pounds, 27" x 4.5" dia $2400 new, $1500 used 1400 Watts	9 oz, 2" x 1.3" dia $26 new ($43 including driver) 900 Watts

1980 vs. 2020 Technology

DC Servo Motor Internal Workings

Internally, a DC motor has

- A permanent magnet (field winding) on the outside,
- An electromagnet on the inside (armature), and
- A commutator which switches the polarity of the current (voltage) so that the torque remains in the same direction as the motor spins.



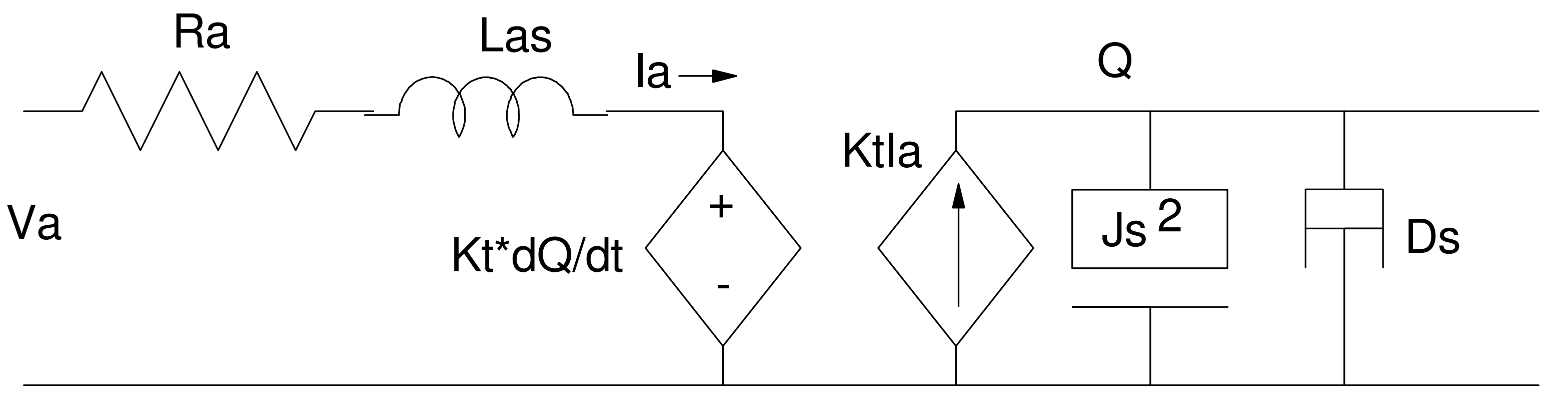
A DC servo motor is essentially an electromagnet

Circuit Equivalent

Kt is the

- The torque constant (Nm / Amp)
- The back EMF (Volts / rad/sec)

MKS: The units are the same (Kt is the same)

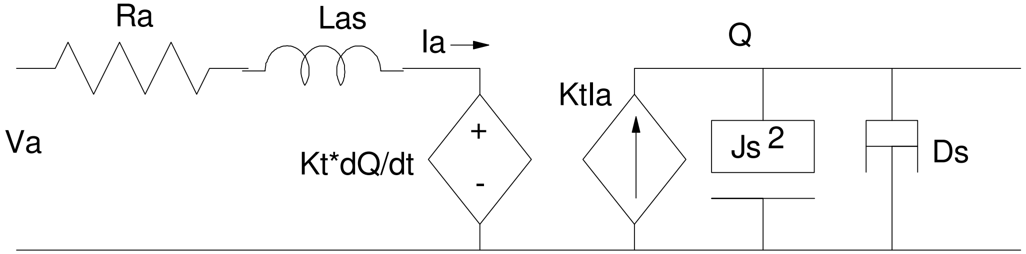


Circuit equivalent for a DC servo motor

Motor Dynamics

$$V_a = K_t(s\theta) + (R_a + L_a s)I_a$$

$$K_t I_a = (Js^2 + Ds)\theta$$



Solving gives

$$\theta = \left(\frac{1}{s}\right)\left(\frac{K_t}{(Js+D)(L_a s+R_a)+K_t^2}\right)V_a$$

$$\omega = \frac{d\theta}{dt} = \left(\frac{K_t}{(Js+D)(L_a s+R_a)+K_t^2}\right)V_a$$

Motor Example:

Electrocraft RDM103

- $K_t = 0.03 Nm/A$
- Terminal resistance = 1.6   (Ra)
- Armature inductance = 4.1mH (La)
- Rotor Inertia = 0.008 oz-in/sec/sec
- Damping Contant = 0.25 oz-in/krpm
- Torque Constant = 13.7 oz-in/amp
- Peak current = 34 amps
- Max operating speed = 6000 rpm

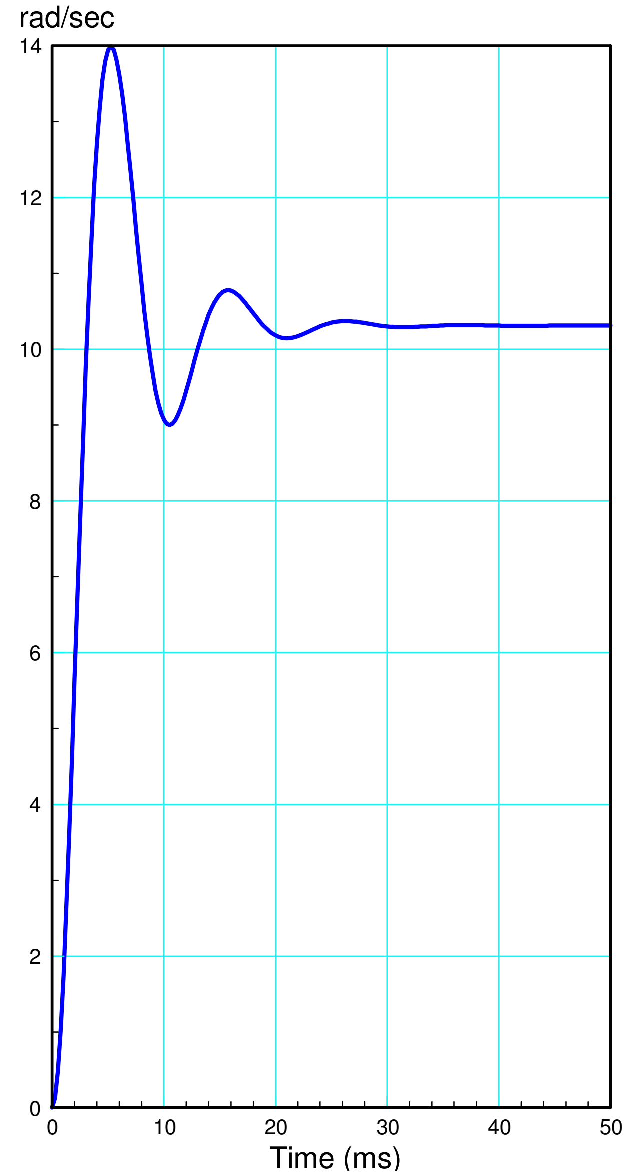
Kt: $$13.7\frac{oz\text{–}in}{amp}\left(\frac{1m}{39.4in}\right)\left(\frac{1lb}{16oz}\right)\left(\frac{0.454kg}{lb}\right)\left(\frac{9.8N}{kg}\right) = 0.0967\left(\frac{N\cdot m}{A}\right)$$

D: $$0.25\left(\frac{oz\cdot in\cdot \min}{krev}\right)\left(\frac{1Nm}{141.7oz\text{–}in}\right)\left(\frac{60\sec}{\min}\right)\left(\frac{krev}{1000rev}\right)\left(\frac{rev}{2\pi rad}\right) = 1.68\cdot 10^{-5}\left(\frac{Nm}{rad/\sec}\right)$$

J: $$0.008\left(\frac{oz\text{–}in}{\sec^2}\right)\left(\frac{Nm}{141.7oz\text{–}in}\right)\left(\frac{kg}{9.8N}\right) = 5.76\cdot 10^{-6}\left(\frac{kg\cdot m}{s^2}\right)$$

Plugging in numbers:

$$\theta = \left(\frac{10.31 \cdot 630^2}{s(s^2+393s+630^2)}\right)V = \left(\frac{10.31(630)^2}{s(s+196.5+j598)(s+196.5-j598)}\right)V$$



This means....

- DC gain = 10.31
  - +12V input makes it spin 123.72 rad/sec
- It takes about 20ms for the motor to get up to speed (Ts = 4/196)

35% overshoot for the step response

- damping ratio = 0.3119
- Note: overshoot goes away if you connect the motor to something
- Inertia increases

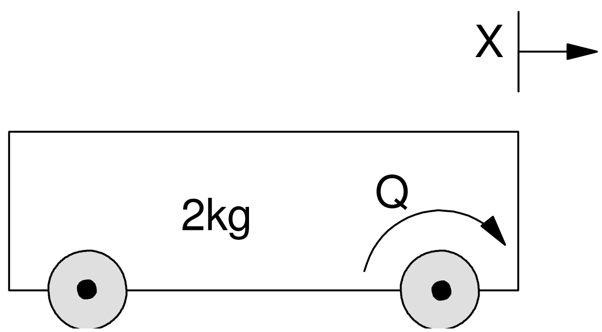
Gears and Wheels:

Wheels are gears

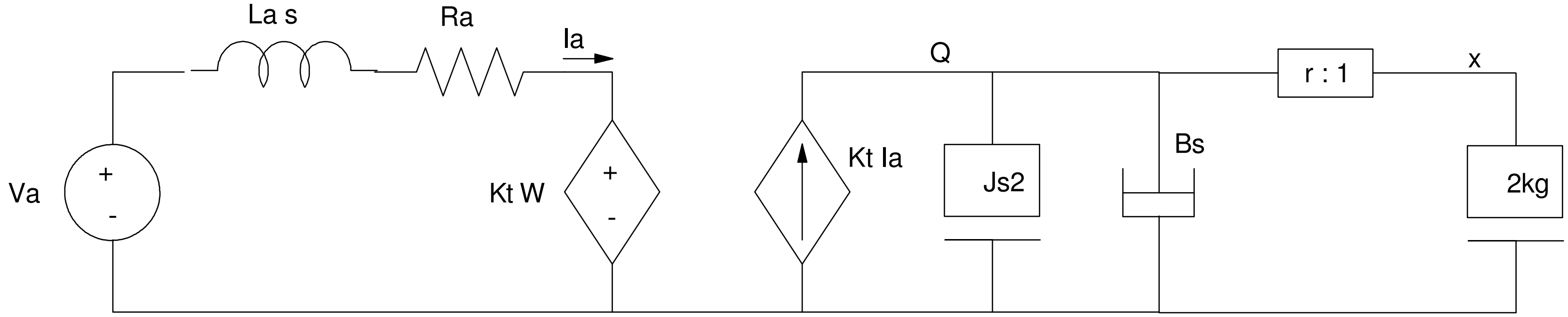
- Convert translation to angle
- $x = r\theta$

Example: Battle Bots

- Find the dynamics of the previous motor if it is driving a cart
- Cart mass = 2kg
- Wheel radius = 1.5cm



First, draw the circuit equivalent



Remove the gear by translating the 2kg mass to the motor angle

$$x = r\theta$$

$$2kg \cdot \left(\frac{r}{1}\right)^2 = 0.0018\ kg\ m^2$$

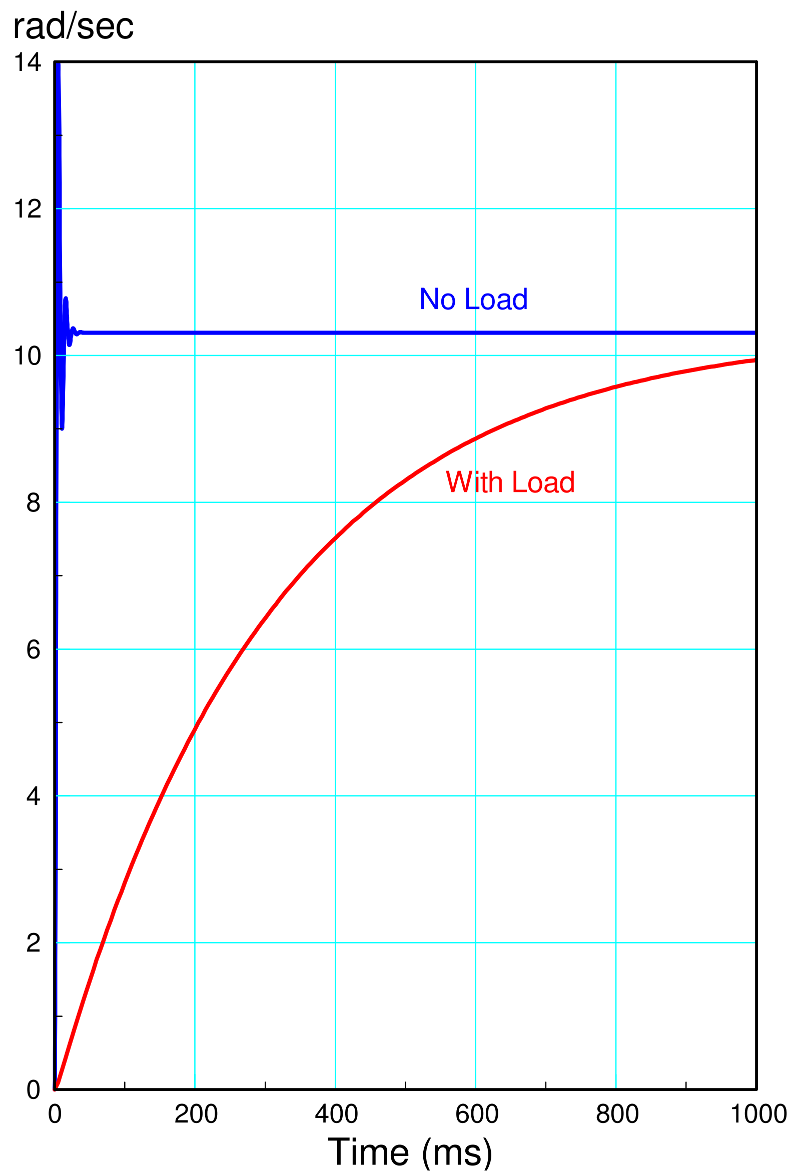
The net inertia is then

$$J_{total} = J_{motor} + J_{mass}$$
$$= 5.76 \cdot 10^{-6} + 0.0018 = 0.00180576$$

The motor dynamics are then

$$\theta = \left(\frac{13061}{s(s+3.273)(s+387)} \right) V_a$$

The added inertia shifted the complex poles



Summary

Rotational systems are similar to translational systems

- Js^2 vs. Ms^2
- Ds vs. Bs
- K vs. K

Drawing the circuit equivalent helps in writing the dynamic equations

Gears act like transformers

- Speed goes through gears as the turns-ratio
- Admittances go through gears as the turn-ratio squared