Mass & Spring Systems & The Wave Equation

ECE 461/661 Controls Systems

Jake Glower - Lecture #15

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Mass & Spring Systems

Each mass has two energy states: Differential equation is of order 2N

- Kinetic Energy
- Potential Energy

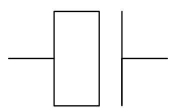
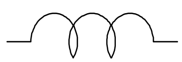
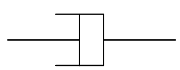
Stategy: (Electrical engineering approach to mass / spring systems)

- Replace the mass, spring, and friction terms with their LaPlace admittance,
- Redraw the system as an electric circuit, and
- Write the voltage node equations.

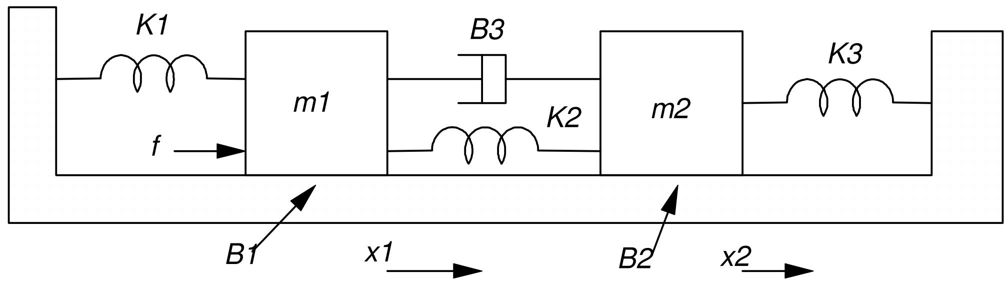
LaPlace Admittances

- Force = Current
- Position = Voltage

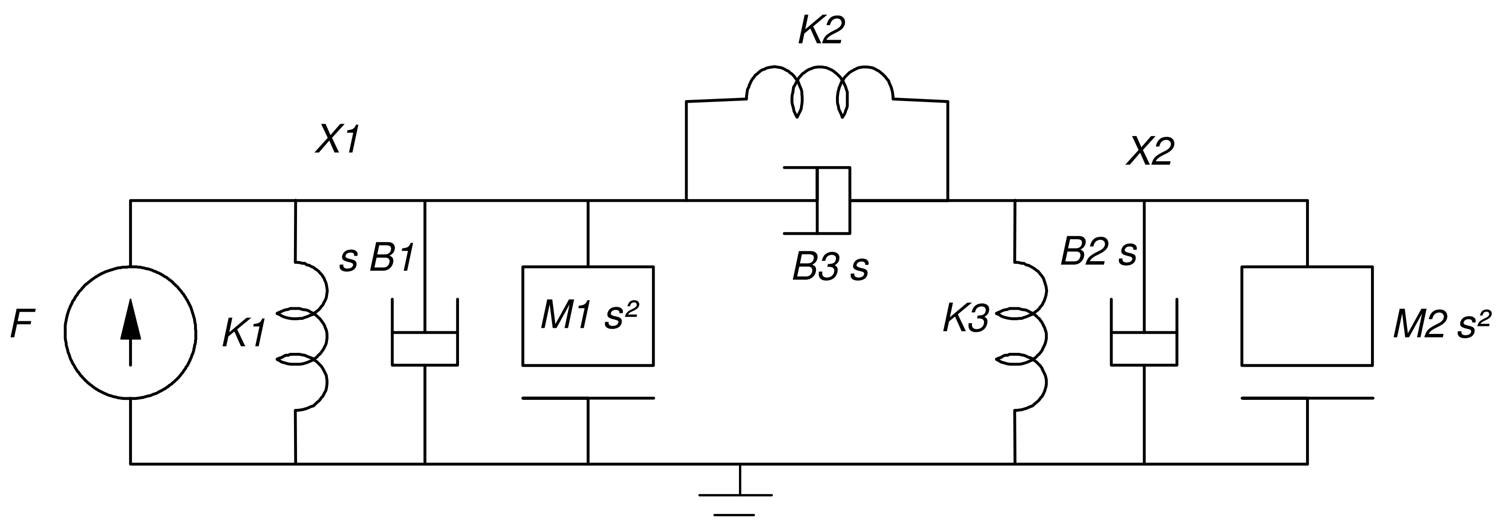
Mechanical World	Electrical World
Force = Mass * Acceleration	Current = Admittance * Voltage

	Symbol	F = ...	LaPlace Addmittance
Mass		f = m x''	s^2 m
Spring		f = k x	k
Friction		f = B x' f = f_v x'	sB s f_v

Example: Mass Spring System



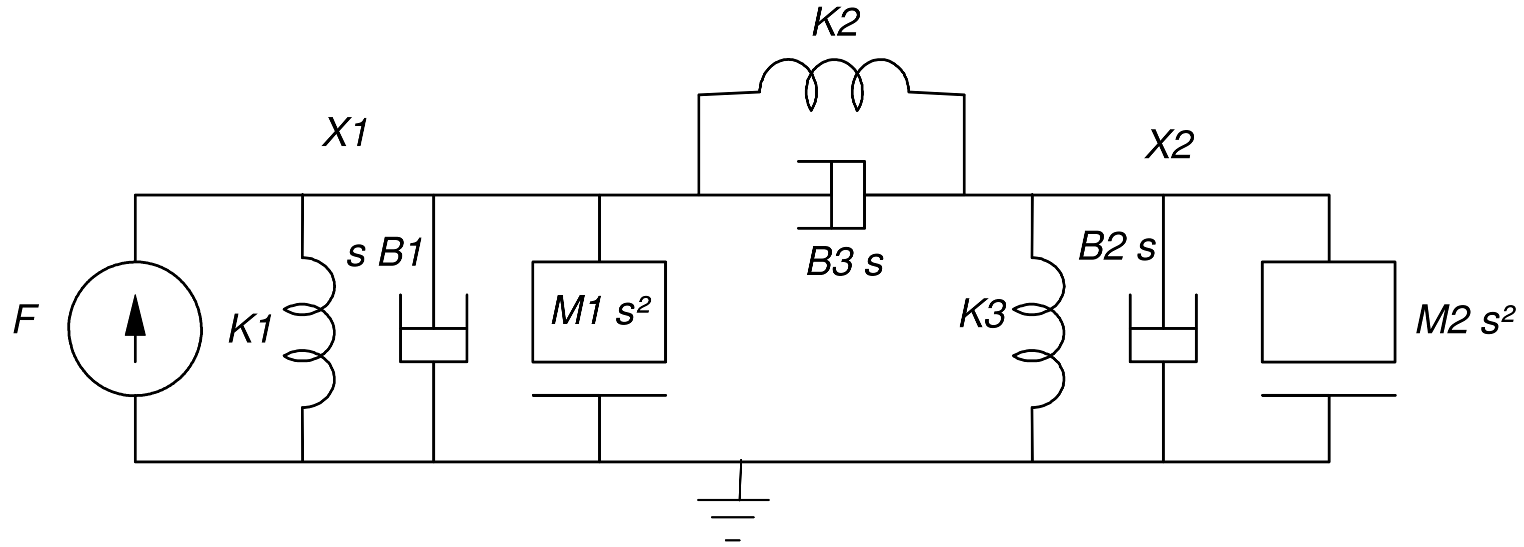
Step 1: Draw the circuit equivalent:



Step 2: Write the voltage node equations

$$(K_1 + B_1 s + M_1 s^2 + K_2 + B_3 s)X_1 - (K_2 + B_3 s)X_2 = F$$

$$(M_2 s^2 + B_2 s + K_3 + K_2 + B_3 s)X_2 - (K_2 + B_3 s)X_1 = 0$$



Step 3: Solve

- hint: use State-Space

Solve for the highest derivative:

$$M_1 s^2 X_1 = -(K_1 + K_2 + B_1 s + B_3 s)X_1 + (K_2 + B_3 s)X_2 + F$$
$$M_2 s^2 X_2 = -(B_2 s + K_3 + K_2 + B_3 s)X_2 + (K_2 + B_3 s)X_1$$

Place in matrix form

$$s\begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ sX_1 \\ sX_2 \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{-(K_1+K_2)}{M_1}\right) & \left(\frac{K_2}{M_1}\right) & \left(\frac{-(B_1+B_3)}{M_1}\right) & \left(\frac{B_3}{M_1}\right) \\ \left(\frac{K_2}{M_2}\right) & \left(\frac{-(K_2+K_3)}{M_2}\right) & \left(\frac{B_3}{M_2}\right) & \left(\frac{-(B_2+B_3)}{M_2}\right) \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ sX_1 \\ sX_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \left(\frac{1}{M_1}\right) \\ 0 \end{bmatrix} F$$

$$Y = X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ sX_1 \\ sX_2 \end{bmatrix} + [0]F$$

Note that

- You have 2N states, where N is the number of masses. Each mass has two energy states (kinetic and potential energy) giving your 2N state variables.
- The first N rows are [0 : I] where I is the identity matrix. This tells MATLAB that the states are position and velocity.
- The last N rows are where the dynamics come into play.

Also also, you can have real or complex poles for mass-spring systems - unlike the heat equation which always has real poles.

Finding the Transfer Function to X2

Assume M = 1kg, B = 2 Ns/m, K =10 N/m

$$s\begin{bmatrix} X_1 \\ X_2 \\ sX_1 \\ sX_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -20 & 10 & -4 & 2 \\ 10 & -20 & 2 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ sX_1 \\ sX_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} F$$

```
A = [0,0,1,0 ; 0,0,0,1 ; -20,10,-4,2 ; 10,-20,2,-4];
B = [0;0;1;0];
C = [0,1,0,0];
D = 0;

G = ss(A,B,C,D)

zpk(G)

              2 (s+5)
-------------------------------
(s^2 + 2s + 10) (s^2 + 6s + 30)
```

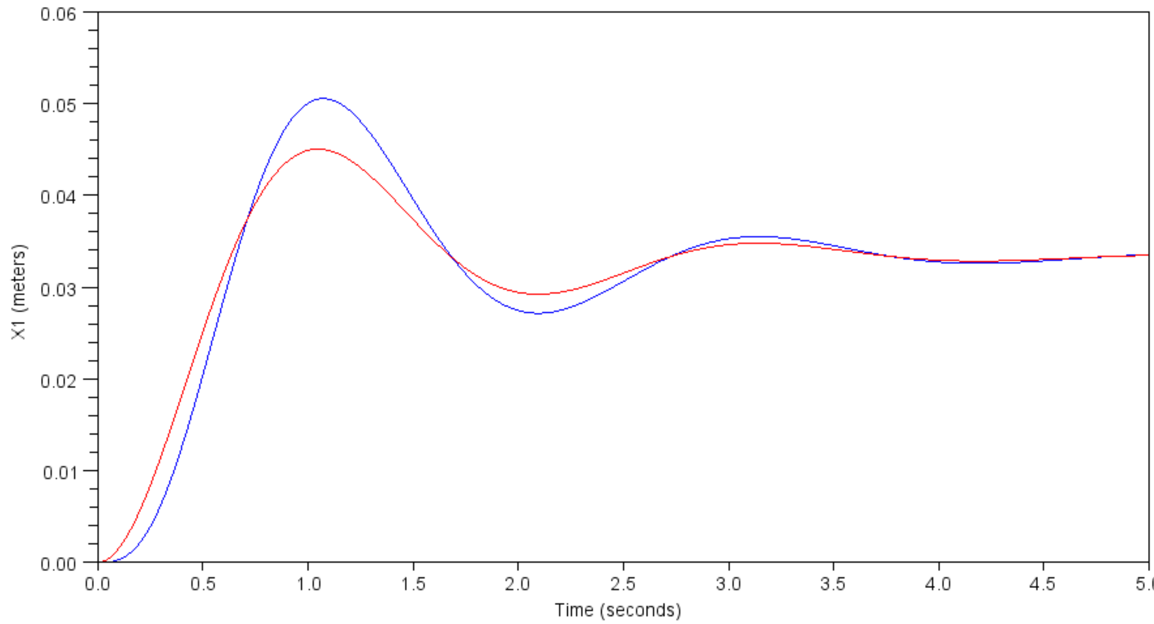
2nd-Order Approximation

- Dominant pole: s = -1 +/- j3
- DC gain = 0.03333

```
DC = evalfr(G,0)
     0.0333333
```

So

$$X_2 \approx \left(\frac{0.3333}{(s+1 \pm j3)} \right) F$$



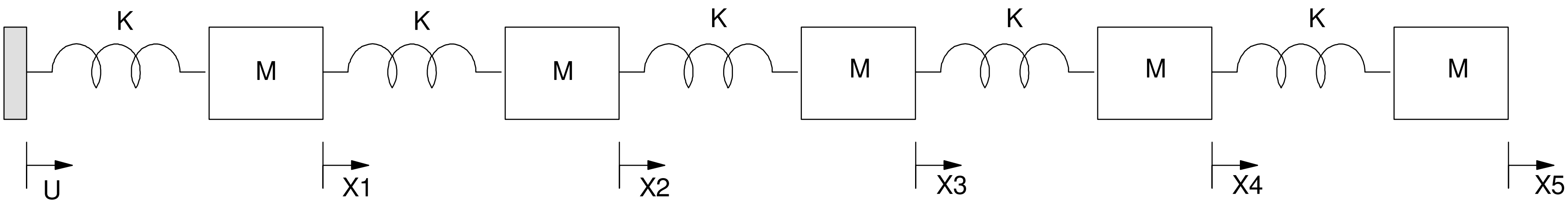
Wave Equation: (fun stuff)

N masses connected by springs:

- Coupled 2nd-order differential equations

$$\frac{d^2 x_i}{dt^2} = f(x_{i-1}, x_i, x_{i+1})$$

- "Wave Equation"



Cascaded Mass-Spring Systems creates the Wave equation

Dynamics:

Node #2

$$Ms^2 x_2 = Kx_1 - 2Kx_2 + Kx_3$$

$$s^2 x_2 = \left(\frac{K}{M}\right)x_1 - \left(\frac{2K}{M}\right)x_2 + \left(\frac{K}{M}\right)x_3$$

Other nodes are similar

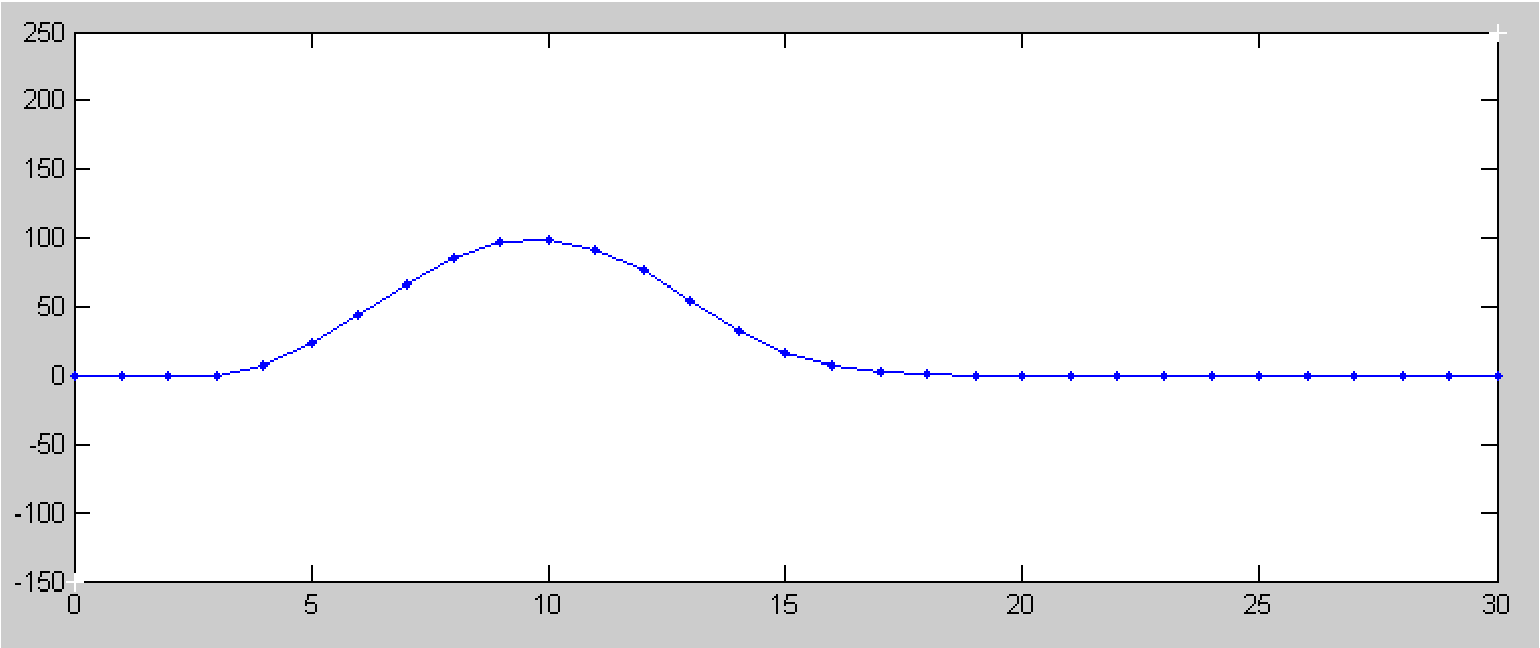
With 30 nodes, you get a 60th order differential equation

- Each node has two energy states
  - Potential Energy
  - Kinetic Energy

30-Node Model

- K/M = 50
- Friction to ground = 0.01
- Snap V0

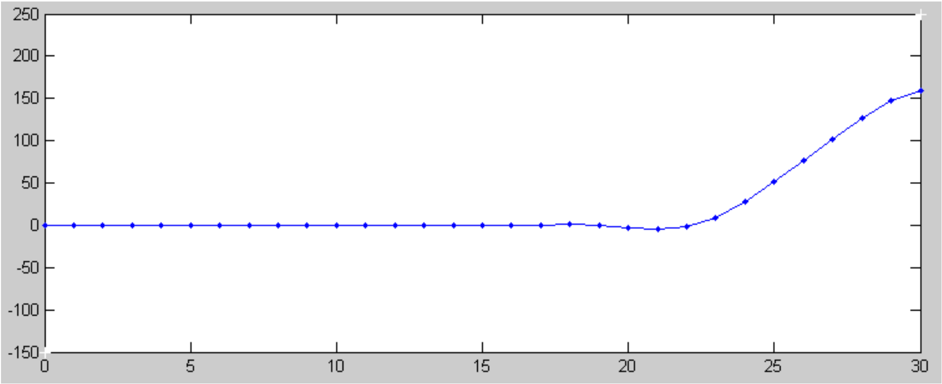
Produces a traveling wave



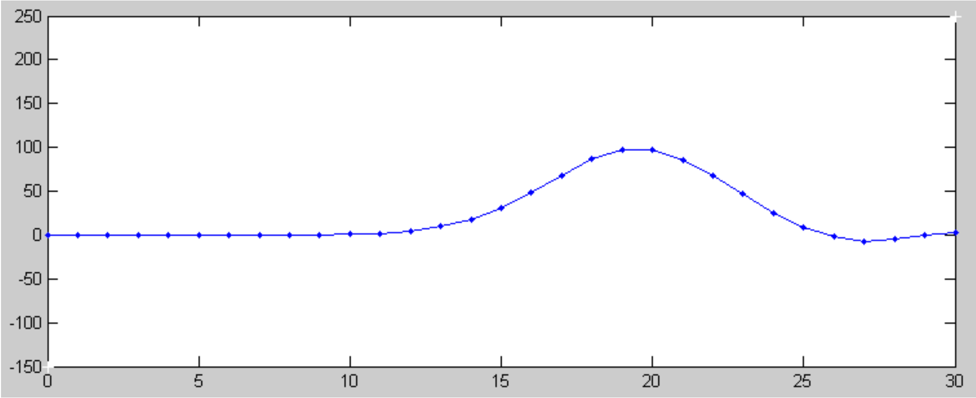
t = 2 seconds. Wave traveling to the right

Reflections:

- Free endpoint causes a + reflection



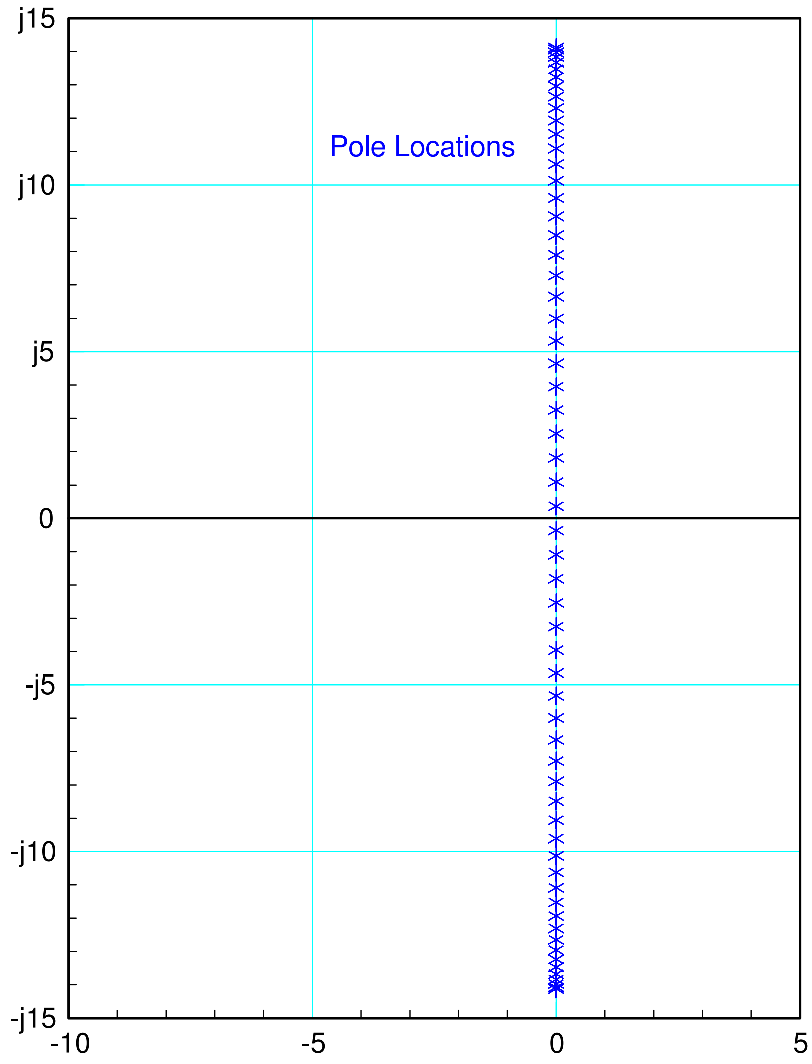
t = 5 seconds. Wave hits the right endpoint



t = 7 seconds. Reflection is now traveling to the left

Really hard system to control

- 60th order system
- All 60 poles are dominant
  - All on the jw axis
  - Scattered from -j14.5 to +j14.5
- 2nd-Order approximations don't work well for this system



Wave.m

```
N = 30;    % number of nodes

V = zeros(N,1);
dV = zeros(N,1);

t = 0;
dt = 0.01;

while(t < 100)

   if (t < 2) V0 = 100 * ( ( sin(0.5*pi*t) )^2 );
      else V0 = 0;
      end

   ddV(1) = 50*V0 - 100*V(1) + 50*V(2) - 0.01*dV(1);

   for i=2:N-1
      ddV(i) = 50*V(i-1) - 100*V(i) + 50*V(i+1) - 0.01*dV(i);
      end

   ddV(N) = 50*V(N-1) - 50*V(N) - 0.01*dV(N);

   for i=1:N
      dV(i) = dV(i) + ddV(i)*dt;
```

```
        V(i) =  V(i) +  dV(i)*dt;
        end
    t = t + dt;

    plot([0:N],[V0;V],'.-');
    ylim([-100,150]);
    pause(0.01);

    end
```