First and Second Order Approximations

ECE 461/661 Controls Systems

Jake Glower - Lecture #11

Please visit Bison Academy for corresponding
lecture notes, homework sets, and solutions

Motivation:

- Each pole is an energy state
- If you have 10 energy states, you get a 10th order transfer function
- Example: 10-stage RC filter with RC = 1

$$Y = \left(\frac{1}{s^{10}+19s^9+153s^8+680s^7+1820s^6+3003s^5+3003s^4+1716s^3+495s^2+55s+1} \right) X$$

We need a model which is

- Simpler, and
- Still fairly accurate
  - Similar step response

Dominant Pole

- One or two poles tend to dominate the response of a system

If you

- Match the dominant pole, and
- Match the DC gain

You get a

- Simpler model (1 or 2 poles),
- Which is fairly accurate (same dominant pole)

Definitions:

Dominant Pole(s):

- The pole which dominates the step response of a system
- The pole which is closest to s=0 (usually)

Transfer Function: G(s)

- The differential equation relating the input (X) and the output (Y)

DC Gain:

- The gain of G(s) at s=0

2% Settling Time:

- The time it takes the transients to decay to 2% of their initial value

Overshoot:

- The maximum of a step response divided by it's steady-state value.

Damping Ratio

- Cosine of the angle of the dominant pole

Which Pole is Dominant?

Example: Find the step response of

$$G(s) = \left(\frac{2000}{(s+1)(s+10)(s+100)}\right)$$

$$Y(s) = \left(\frac{2000}{(s+1)(s+10)(s+100)}\right)\left(\frac{1}{s}\right)$$

$$Y(s) = \left(\frac{2}{s}\right) + \left(\frac{-2.222}{s+1}\right) + \left(\frac{0.2469}{s+10}\right) + \left(\frac{0.0022}{s+100}\right)$$

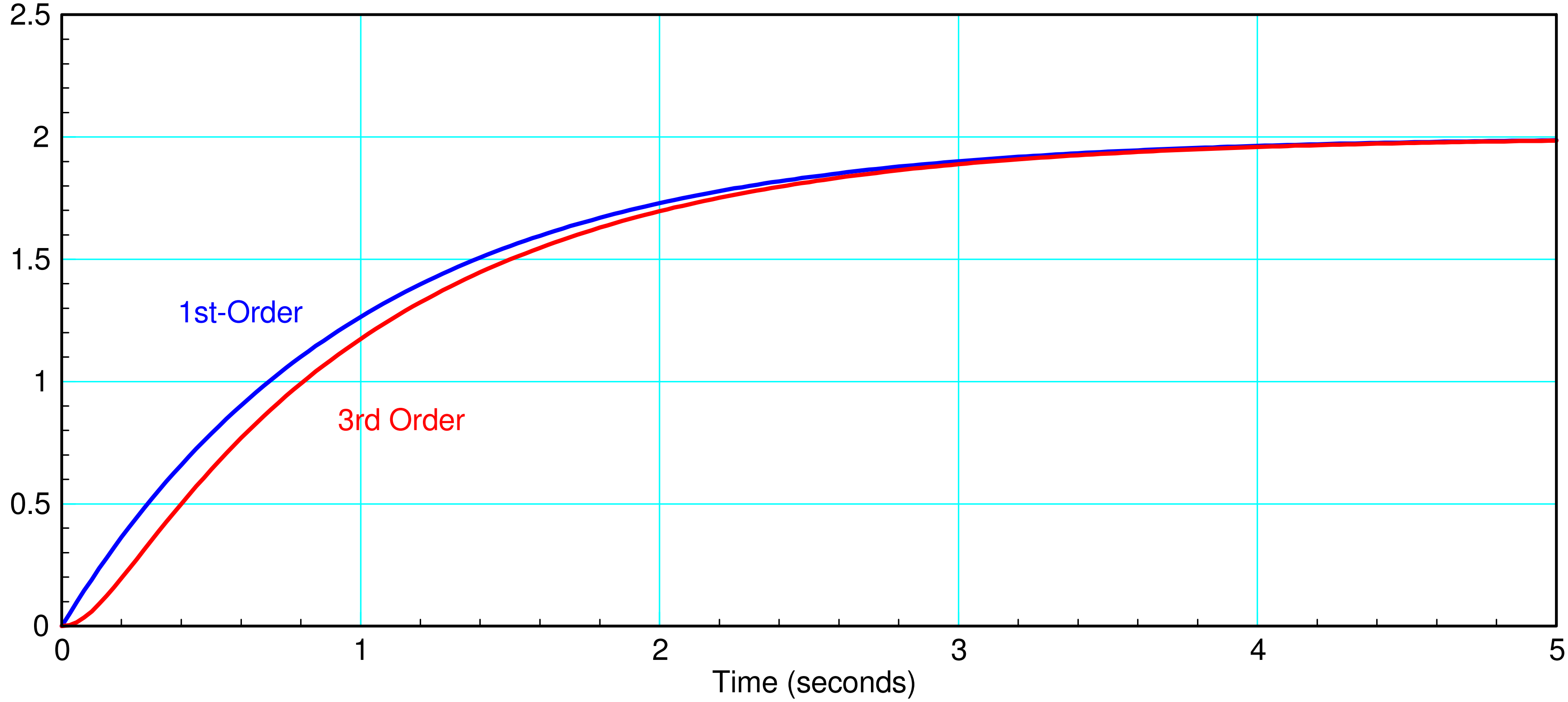
$$y(t) = (2 - 2.222e^{-t} + 0.2469e^{-10t} - 0.0022e^{-100t})u(t)$$

Note that

- The DC gain is 2 (first term)
- The second term dominates the transient response
  - Larger than the other terms
  - Lasts longer than the other terms

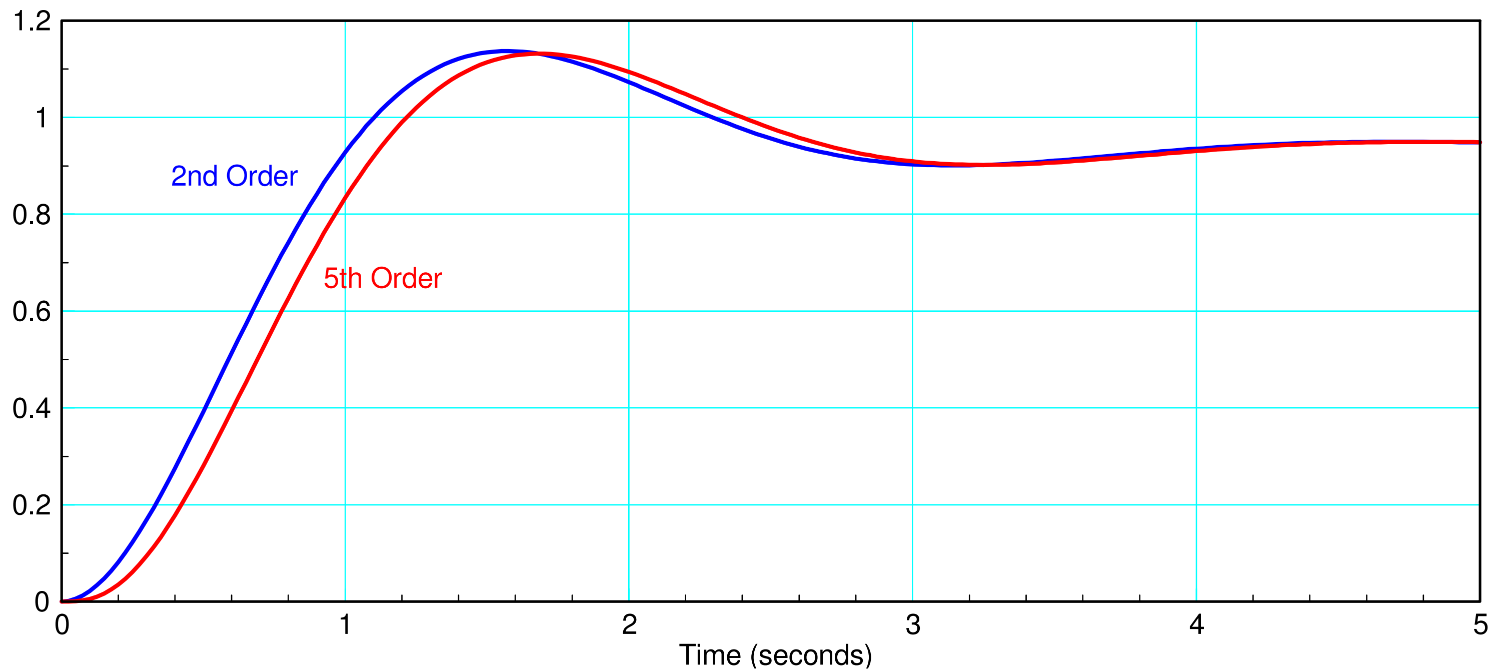
Real Dominant Pole: $G(s) = \left(\frac{2000}{(s+1)(s+10)(s+100)} \right) \approx \left(\frac{2}{s+1} \right)$

- Keep the dominant pole (s = -1)
- Match the DC gain (2)



Complex Dominant Poles:

- $G(s) = \left(\frac{2{,}000{,}000}{(s+1+j2)(s+1-j2)(s+10)(s+50+j200)(s+50-j200)} \right) \approx \left(\frac{4.70588}{(s+1+j2)(s+1-j2)} \right)$
- Keep the poles at -1 +/- j2
- Match the DC gain



Time Scaling

In this class, the dominant pole is usually close to 1.000

- Good numerical properties

This implies time scaling

- The x-axis could be milliseconds rather than seconds

LaPlace transforms assume all functions are of the form

$$y = \exp(st)$$

If you change time from seconds to milliseconds

$$\tau = 1000t$$

the pole becomes 1000x smaller

$$y = \exp\left(\frac{s}{1000}\,1000t\right) = \exp\left(\frac{s}{1000}\,\tau\right)$$

First-Order Approximations:

- Describe the step response by inspection
- Look at the dominant pole and the DC gain

Generic 1st-order system: Two degrees of freedom

$$G(s) = \left(\frac{a}{s+b}\right)$$

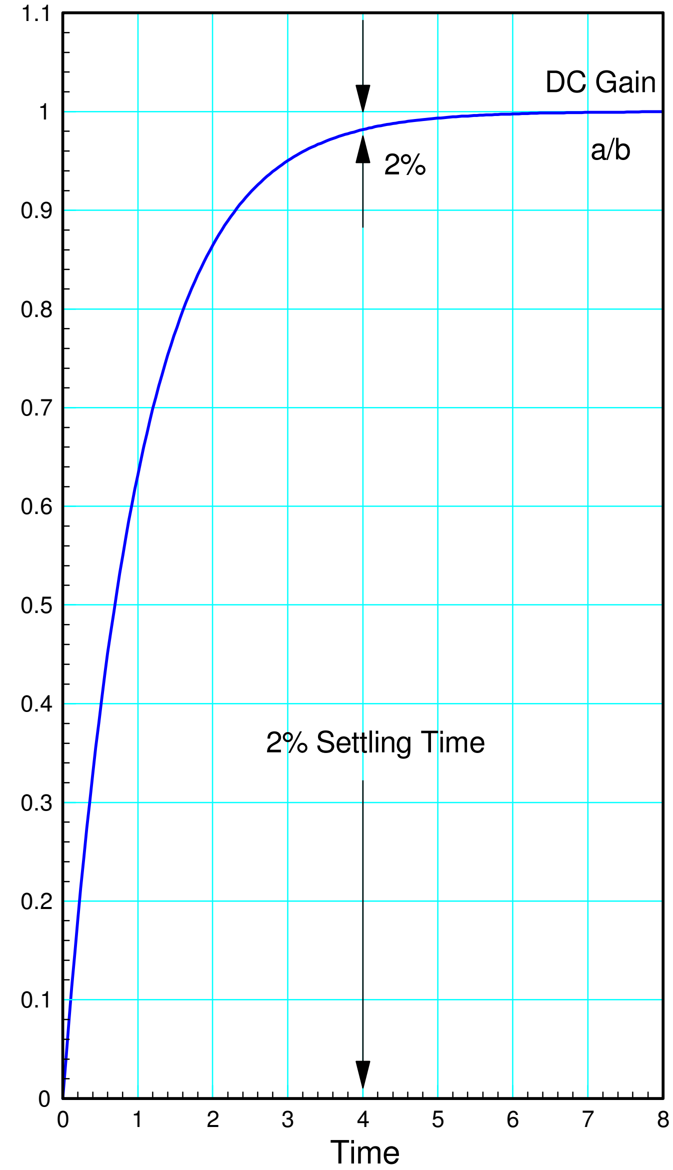
DC Gain: The DC gain is the gain at s = 0:

$$DC = \left(\frac{a}{b}\right)$$

2% Settling Time:

$$e^{-bt} = 0.02$$

$$t = \frac{4}{b}$$



Example: 10th Order System

Sketch the step response of:

$$Y=\left(\frac{1}{s^{10}+19s^{9}+153s^{8}+680s^{7}+1820s^{6}+3003s^{5}+3003s^{4}+1716s^{3}+495s^{2}+55s+1}\right)X$$

Solution: Find the DC gain

```
DC = evalfr(G10,0)
    1.0000
```

$$DC=\left(\frac{1}{s^{10}+19s^{9}+153s^{8}+680s^{7}+1820s^{6}+3003s^{5}+3003s^{4}+1716s^{3}+495s^{2}+55s+1}\right)_{s=0}=1$$

Find the dominant pole & the 2% settling time:

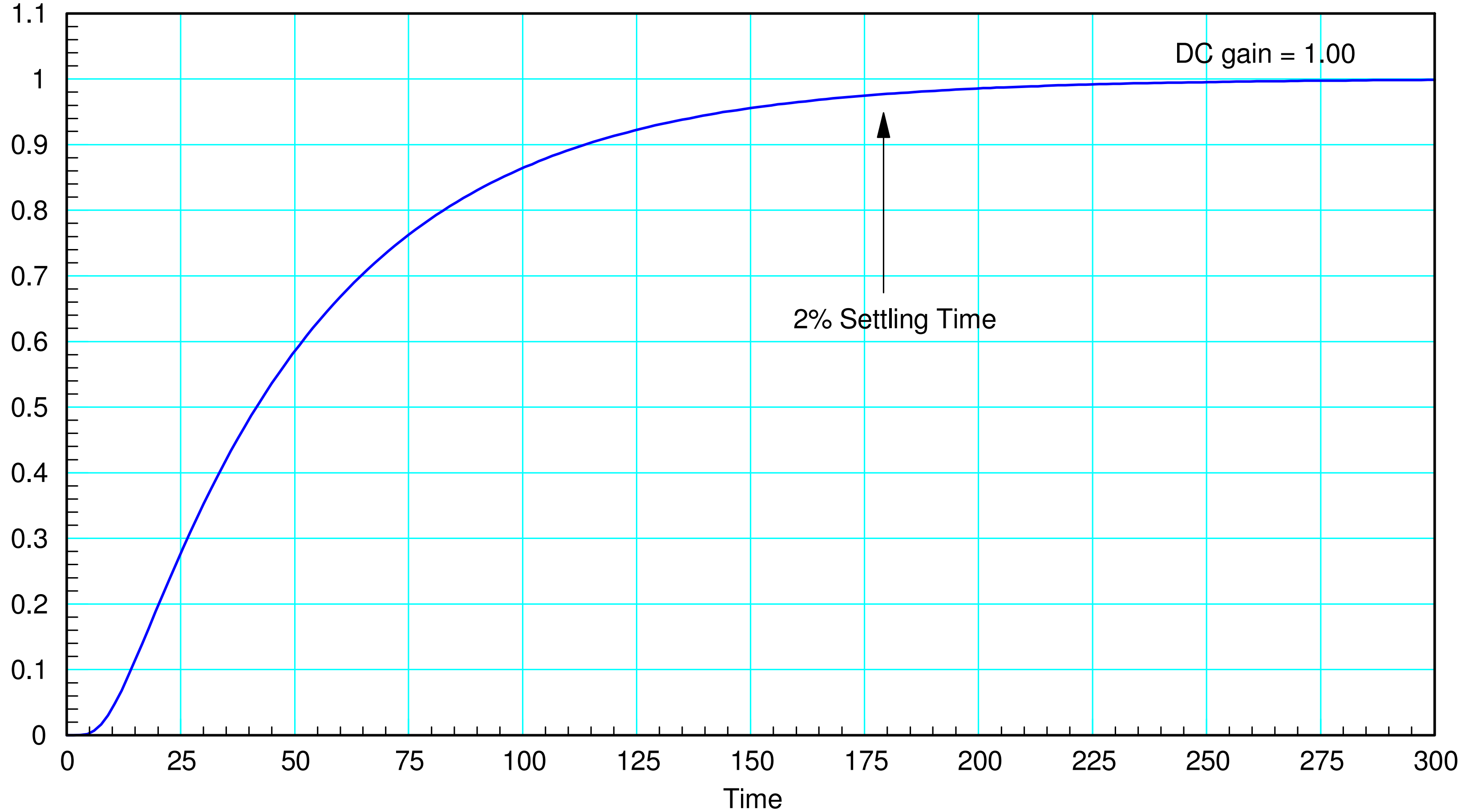
```
zpk(G10)

                                                    1
-----------------------------------------------------------------------------------------------------------
(s+3.911) (s+3.652) (s+3.247) (s+2.731) (s+2.149) (s+1.555) (s+1) (s+0.5339) (s+0.1981) (s+0.02234)
```

$$t_{2\%}=\frac{4}{0.02234}=179.05\text{ seconds}$$

Checking in Matlab:

```
>> t = [0:0.1:300]';
>> y10 = step(G10,t);
>> plot(t,y10);
```



As expected, the 10th-Order RC Filter has a DC gain of one and a 2% settling time of about 179 seconds

Going Backwards:

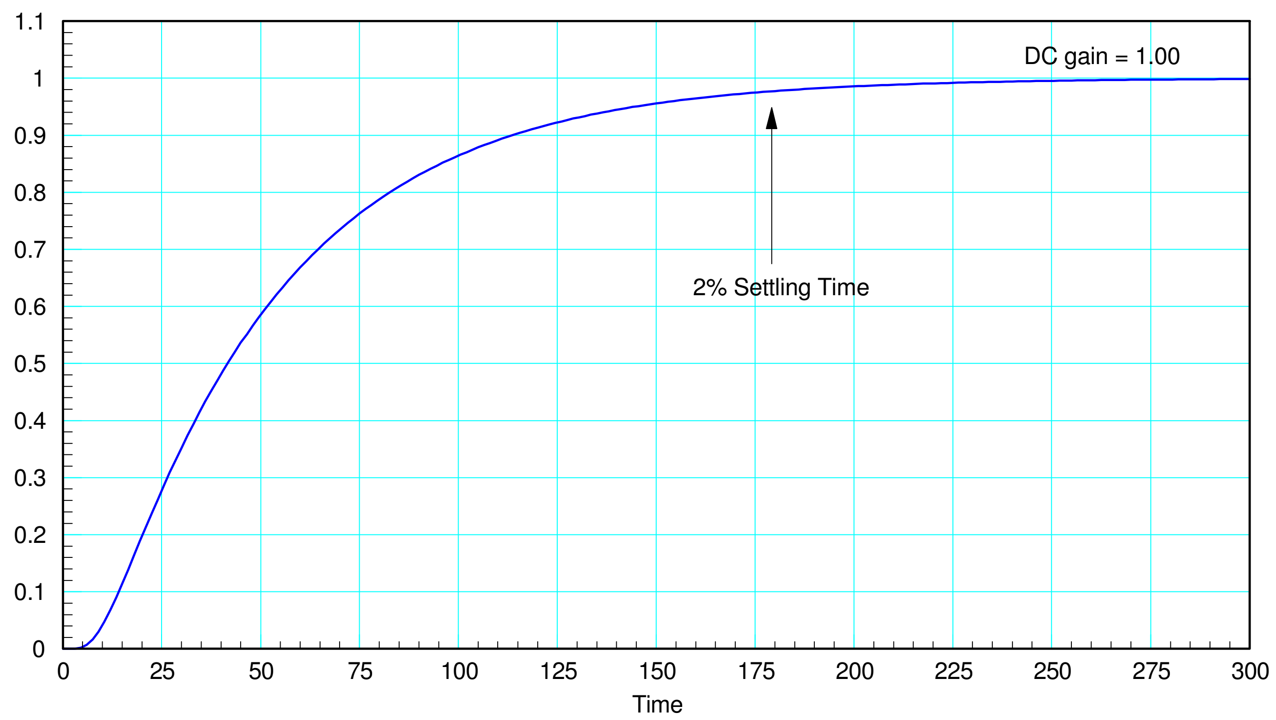
Give the step response, find G(s)

- The DC gain is 1.00
- The 2% settling time is 178 seconds (approximately).

This tells you that the dominant pole is

$$b \approx \frac{4}{178} = 0.0225$$

$$G(s) \approx \left(\frac{0.0225}{s+0.0225}\right)$$

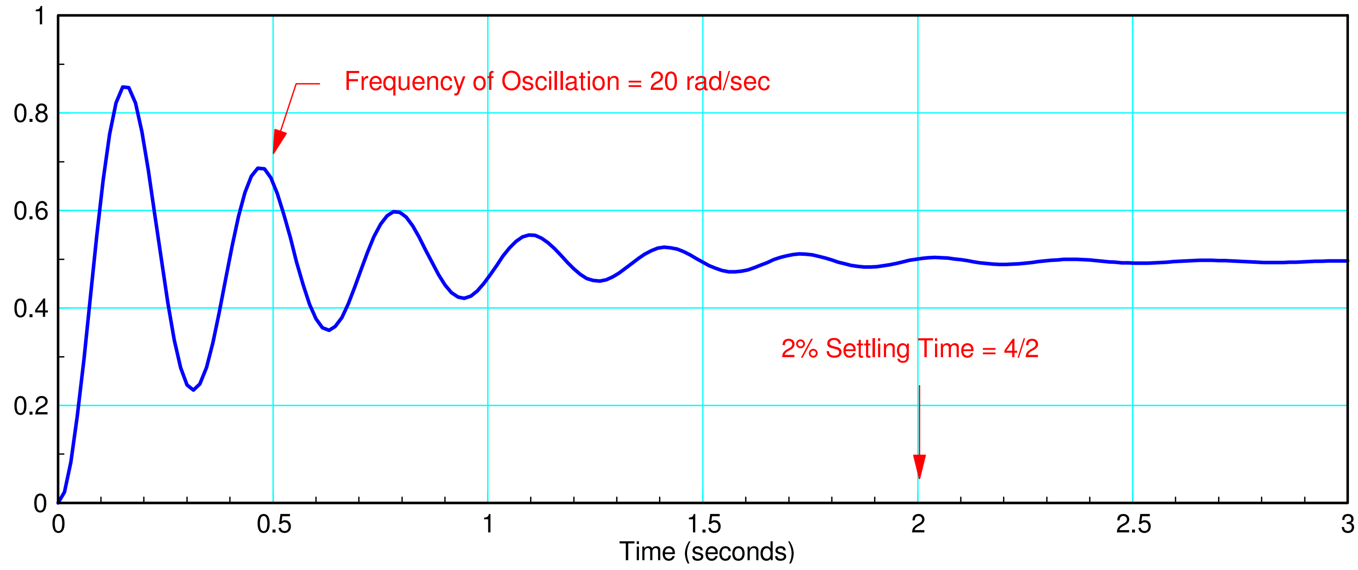


Second-Order Approximations:

$$G(s) = \left(\frac{ac}{s^2+bs+c}\right) = \left(\frac{a\omega_n^2}{(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}\right)$$

- **The real part of the dominant pole determines the 2% settling time**
- **The complex part of the pole determines the frequency of oscillation**

Example: $G(s) = \left(\frac{200}{(s+2+j20)(s+2-j20)}\right)$



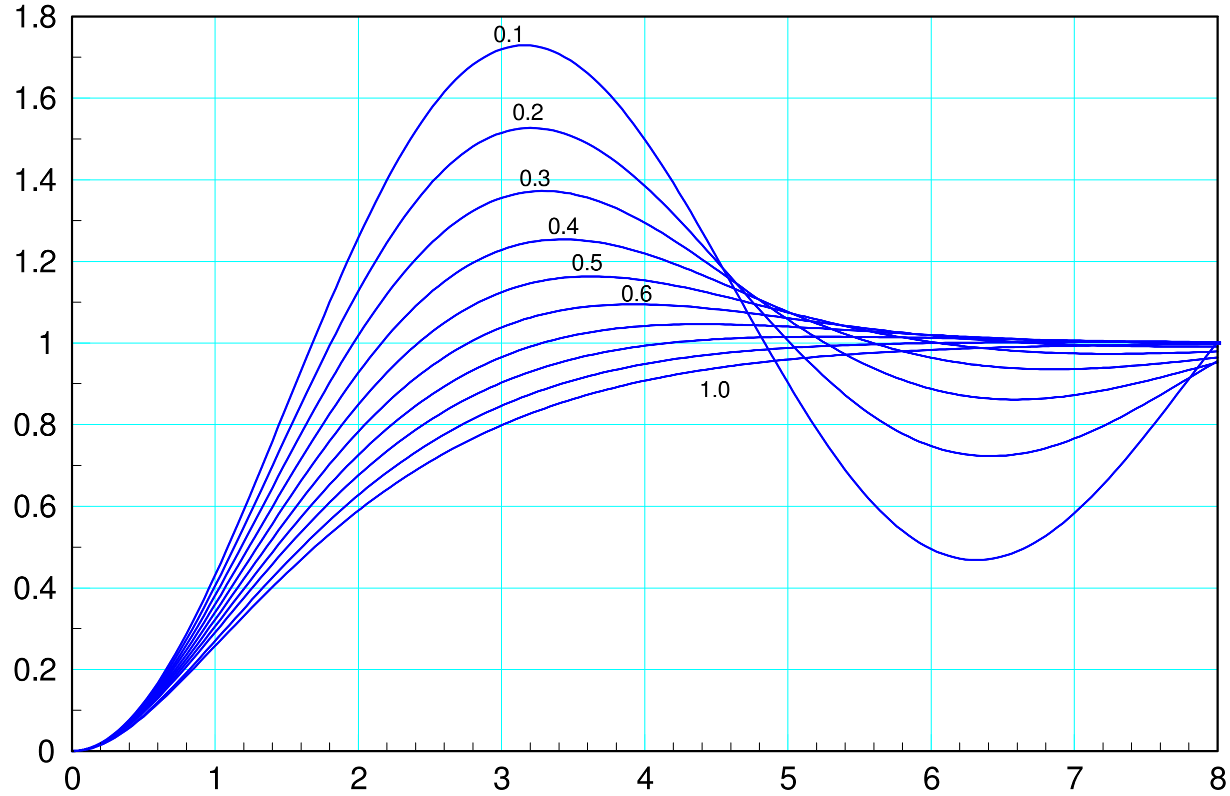
Second Order Approximations (take 2)

$$G(s) = \left(\frac{a\omega_n^2}{(s+\omega_n\angle\theta)(s+\omega_n\angle-\theta)}\right) = \left(\frac{a\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

The angle tells you the % Overshoot

- $\zeta = \cos\theta$ = damping ratio
- $OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

zeta	Overshoot
1.0	0.00%
0.9	0.15%
0.8	1.52%
0.7	4.60%
0.6	9.48%
0.5	16.30%
0.4	25.38%
0.3	37.23%
0.2	52.66%
0.1	72.92%
0.0	100%



Example: Determine the step response of the following system by inspection:

$$G(s) = \left(\frac{2000}{(s+1+j2)(s+1-j2)(s+10)(s+50+j200)(s+50-j200)}\right)$$

Solution: The DC gain is 0.94

The dominant pole is at -1 + j2

- The 2% settling time will be 4 seconds (4/1)
- The frequency of oscillation will be 2 rad/sec
- The overshoot will be 20.79%

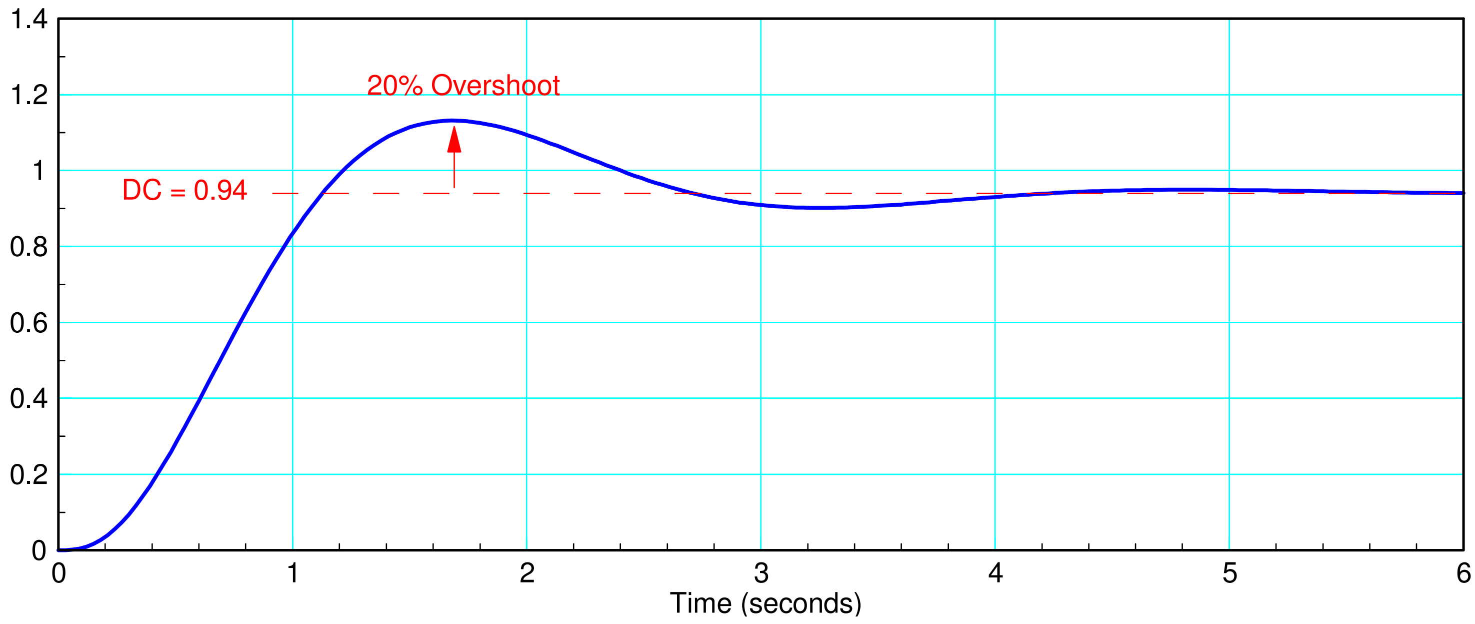
$$1 + j2 = 2.23\angle 63.4^0$$

$$\zeta = \cos(63.4^0) = 0.447$$

$$OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 20.79\%$$

The actual step response is:

```
G = zpk([],[-1+j*2,-1-j*2,-10,-50+j*200,-50-j*200],2e6);
t = [0:0.001:6]';
y = step(G,t);
DC = evalfr(G,0)
DC =  0.9412
plot(t,y,'b',[0,6],[1,1]*DC,'c--')
OS = max(y) / DC
OS =  1.2021
```



Finding G(s) from the step response

- DC gain = 0.94
- 2% Settling time = 2 seconds
- 20% overshoot

$$\zeta = 0.4536$$

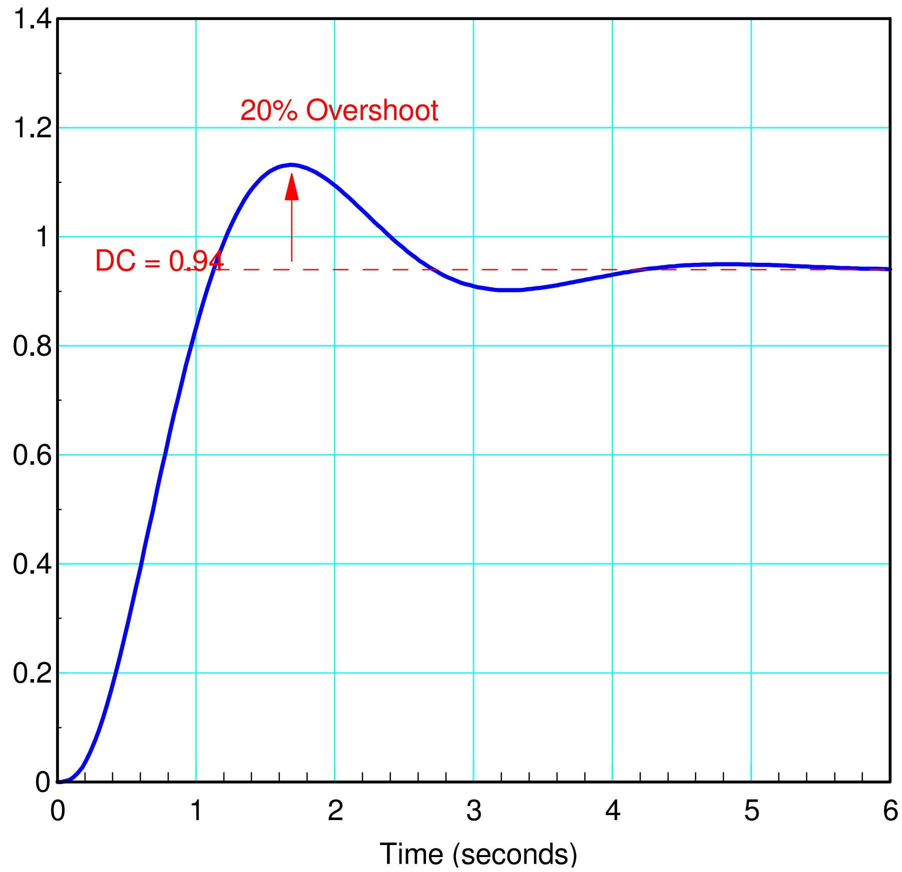
$$\theta = \arccos(\zeta) = 63.02^0$$

$$s = -1 + j1.96$$

$$G(s) \approx \left(\frac{a}{(s+1+j1.96)(s+1-j1.96)} \right)$$

$$G(s) \approx \left(\frac{4.55}{(s+1+j1.96)(s+1-j1.96)} \right)$$

note: you only find the dominant pole



Summary

Usually, a transfer function has one or two poles that dominate the step response

- One: a real pole
- Two: a complex conjugate pair of poles

By looking at the dominant pole(s), you can pretty much tell how the system will behave.

- The other fast poles don't really matter that much

When you specify how the system should behave, you're actually specifying where the dominant pole(s) belong

- Again, the other fast poles don't really matter that much