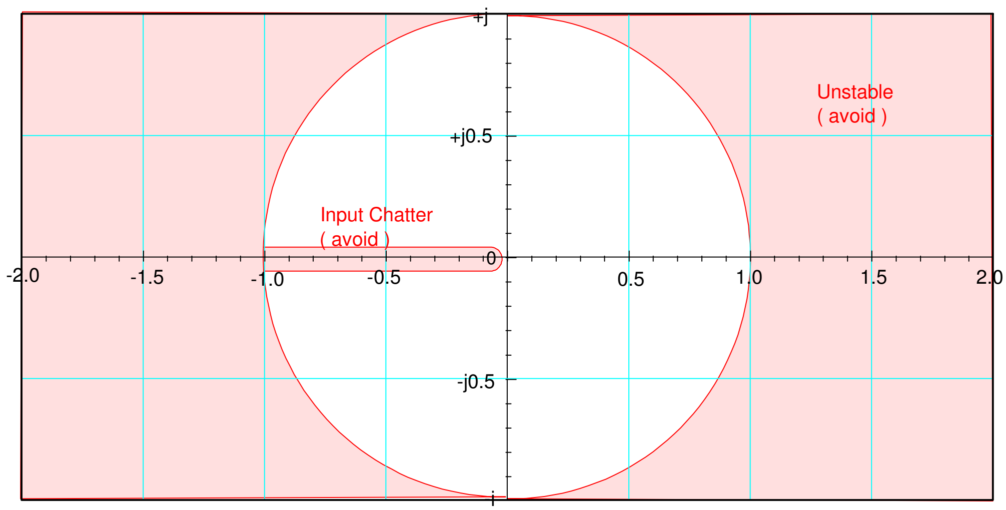
Meeting Design Specs in the z-plane

Like the s-plane, you can add poles and zeros to a compensator in the z-plane to meet design specs. In general

- You add zeros to cancel slow poles
- For every zero you add, you need to add a pole
- One of these poles goes to s=0 (z=1) if you need to make the system type-1, and
- The remaining poles go somewhere out of the way (or are adjusted to meet the design specs).

There are a few things to avoid when designing a digital compensator, however.

- Avoid placing poles outside the unit circle. This results in the open-loop system being unstable, which makes it difficult (sometimes dangerous) to test and debug.
- Avoid placing poles on the negative real axis (between -1 and 0). This results in the input switching between positive and negative each sample, which tends to wear out actuators.



Regions to avoid for placing the poles of K(z) when designing a digital compensator

Example: Design a compensator, K(z), for the following system

$$G(s) = \left(\frac{50}{(s+1)(s+3)(s+10)}\right)$$

that results in

- No error for a step input,
- 20% overshoot for a step input, and
- A 2% settling time of 4 seconds.

Method #1: z-Plane Analysis

Since the sampling rate was not specified, pick T. Since the settling time is 4 seconds, a reasonable sampling rate is 200ms

T = 0.2

This gives the controller twenty iterations to force the output to track the input. (perhaps a bit much)

Since K is in the z-plane, convert everything to the z-plane. From previous methods

$$G(z) \approx \left(\frac{0.1179z}{(z-0.8187)(z-0.5488)(z-0.1353)} \right)$$

The design requirements translate to

- Make the system type-1
- Place the closed-loop dominant pole at s = -1 + j2 (s-plane)
- Place the closed-loop dominant pole at z = 0.7541 + j0.3188 (z-plane found from $z = e^{sT}$)

Let

$$K(z) = k\left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-a)} \right)$$

$$GK = \left(\frac{0.1179kz}{(z-1)(z-0.1353)(z-a)} \right)$$

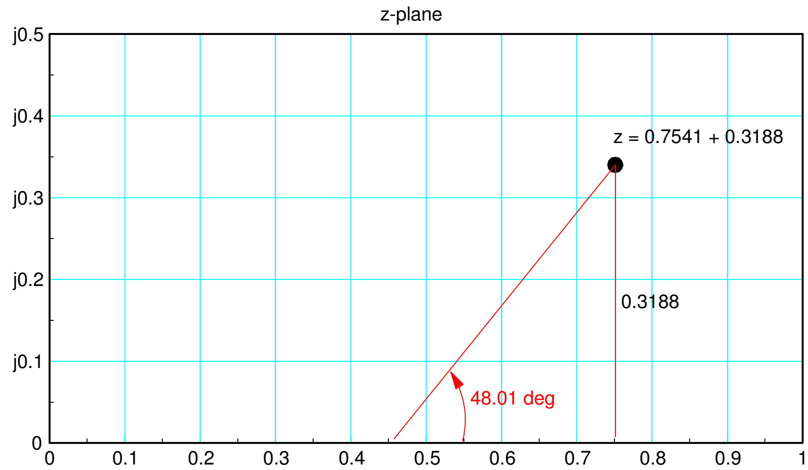
Pick 'a' so that 0.7541 + j0.3188 is on the root locus. Taking the part you know:

$$\left(\frac{0.1179z}{(z-1)(z-0.1353)} \right)_{z=0.7541+j0.3188} = 0.3444\angle -131.98^0$$

meaning (z-a) must contribute -48.01 degrees to make the angles add up to 180 degrees. 'a' is then

$$a = 0.7541 - \left(\frac{0.3188}{\tan\left(48.01^0\right)} \right)$$

$$a = 0.4672$$



To find 'k', set GK = -1 at z = 0.7541 + j0.3188

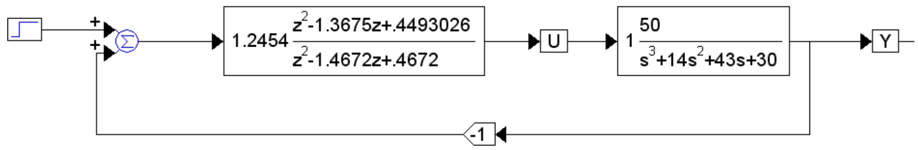
$$\left(\frac{0.1179z}{(z-1)(z-0.4672)(z-0.1353)}\right)_{z=0.7541+j0.3188} = 0.8029\angle 180^0$$

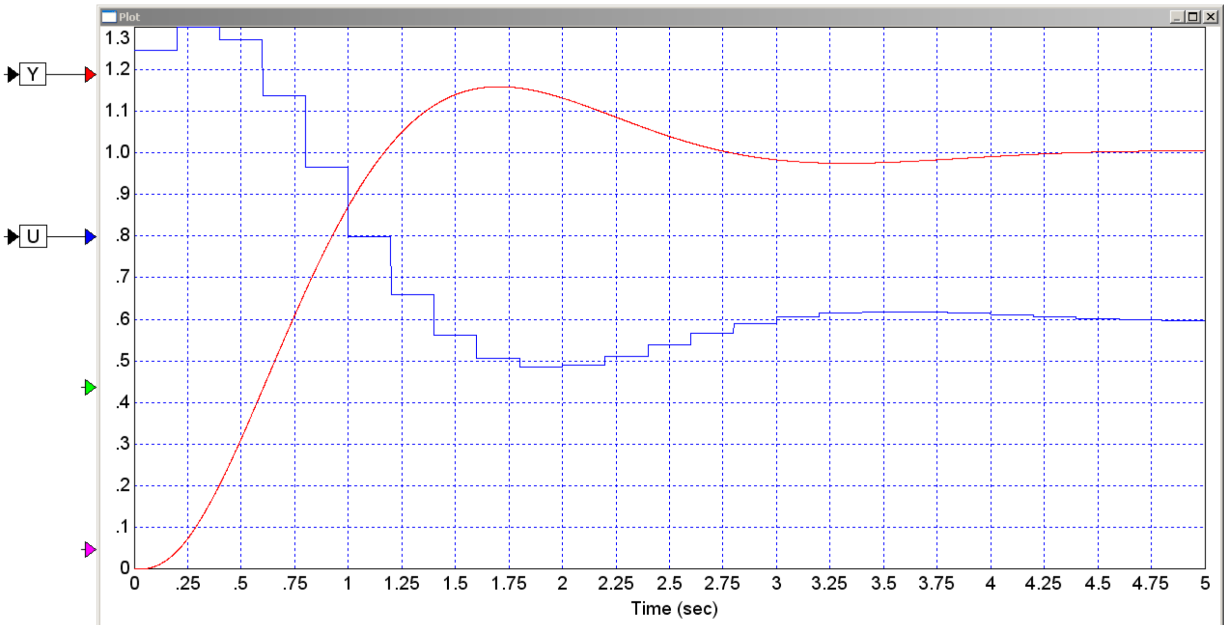
$$k = \frac{1}{0.8029} = 1.2454$$

and

$$K(z) = 1.2454\left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-0.4672)}\right)$$

Checking in VisSim:





Note that the overshoot is a little off. This is due to the model for G(z) being slightly off.

Method #2: Mixed Analysis

Since we really don't need to sketch the root locus, you don't really need to convert G(s) to G(z). All you really need is to be able to analyze G(s) and K(z) at the same point. The conversion $z = e^{sT}$ allows you to do this.

Step 1: Decide where you want to place the closed-loop poles. From before

- s = -1 + j2
- z = 0.7541 + j0.3188

Step 2: Model G(s) and the zero-order hold (modeled as a 1/2 sample delay)

$$G(s) \cdot ZOH = \left(\frac{50}{(s+1)(s+3)(s+10)}\right) \cdot e^{-sT/2}$$

Step 3: Pick the form of K(z)

$$K(z) = k\left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-a)}\right)$$

$$G \cdot K \cdot ZOH = \left(\frac{50}{(s+1)(s+3)(s+10)}\right) \cdot e^{-sT/2} \cdot k\left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-a)}\right)$$

To find 'a', evaluate at s (z). Pick 'a' to make the angles add up to 180 degrees

$$\left(\left(\frac{50}{(s+1)(s+3)(s+10)}\right) \cdot e^{-sT/2} \cdot \left(\frac{(z-0.8187)(z-0.5488)}{(z-1)}\right)\right)_{s=-1+j2} =$$

$$= ((0.9587\angle-147^0) \cdot (0.9048\angle-11.46^0) \cdot (0.3063\angle 31.03^0))$$

$$= 0.2657\angle-127.96^0$$

To make the angle 180 degrees, (z-a) contributes 52.04 degrees

$$a = 0.7541 - \left(\frac{0.3188}{\tan\left(52.04^0\right)}\right) = 0.5054$$

and

$$K(z) = k\left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-0.5054)}\right)$$

To find 'k'

$$\left(\left(\frac{50}{(s+1)(s+3)(s+10)}\right) \cdot e^{-sT/2} \cdot \left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-0.5054)}\right)\right)_{s=-1+j2} = 0.8028\angle 180^0$$

so

$$k = \frac{1}{0.8028}$$

and

$$K(z) = 1.2456\left(\frac{(z-0.8187)(z-0.5488)}{(z-1)(z-0.5054)}\right)$$

Checking in VisSim: The results is much closer to 20% overshoot due to using G(s) rather than approximating it with G(z)