Meeting Design Specs using Root Locus

So far, we have

- Lead compensators which cancel a pole and move it left, speeding up the root locus.
- PID compensators which add a zero at s=0 and add zero, one, or two zeros, pulling the root locus left, speeding up the system.

Essentially, what you're doing is

- Adding a pole at s = 0 (if needed) to make the system type-1
- Adding zeros to cancel poles, speeding up the system, and
- Adding poles that are left and out of the way so that the compensator, K(s), has finite high-frequency gain.

If you don't care whether the resulting K(s) has a name or not, then you're free to add zeros and poles as needed. This lets you design a compensator to meet some design specifications.

Example: Design a controller for the following system

$$ G(s) = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)} \right) $$

so that the closed-loop system meets the following requirements

- No error for a step input,
- A 2% settling time of 4 seconds,
- 20% overshoot for a step input, and
- The high-frequency gain of K(s) is finite

Solution: First, translate the requirements to root-locus terms.

- No error means the system should be type-1. It's presently type-0, so add a pole at s = 0
- The 2% settling time tells you that the real part of the closed-loop dominant pole should be at -1
- The overshoot tells you that the damping ratio should be 0.4559
- A finite high-frequency gain means the number of poles is greater than or equal to the number of zeros in K(s)

In short

Make the system type-1

Place the closed-loop dominant pole at s = -1 + j2

Step 1) Add a pole at s = 0 to make the system type-1

$$ K(s) = \left(\frac{k}{s} \right) $$

Step 2) Start canceling poles until the system is too fast (the root locus passes left of -1 + j2)

$$K(s) = k\left(\frac{(s+0.3234)(s+2.081)}{s}\right)$$

Step 3) Since you added two zeros, you need to add two poles (finite high frequency gain). One pole belongs at s = 0. The other is unknown at present

$$K(s) = k\left(\frac{(s+0.3234)(s+2.081)}{s(s+a)}\right)$$

Pick 'a' so that s = -1 + j2 is on the root locus. To do this, pick 'a' so that the angles add up to 180 degrees

$$GK = \left(\frac{361.2378}{(s+0.3234)(s+2.081)(s+5.439)(s+10.1)(s+15.65)}\right)\left(\frac{k(s+0.3234)(s+2.081)}{s(s+a)}\right)$$

$$GK = \left(\frac{361.2378k}{s(s+5.439)(s+10.1)(s+15.65)(s+a)}\right)_{s=-1+j2} = 1\angle180^0$$

Evaluate what you know:

$$\left(\frac{361.2378}{s(s+5.439)(s+10.1)(s+15.65)}\right)_{s=-1+j2} = 0.2409\angle-160.99^0$$

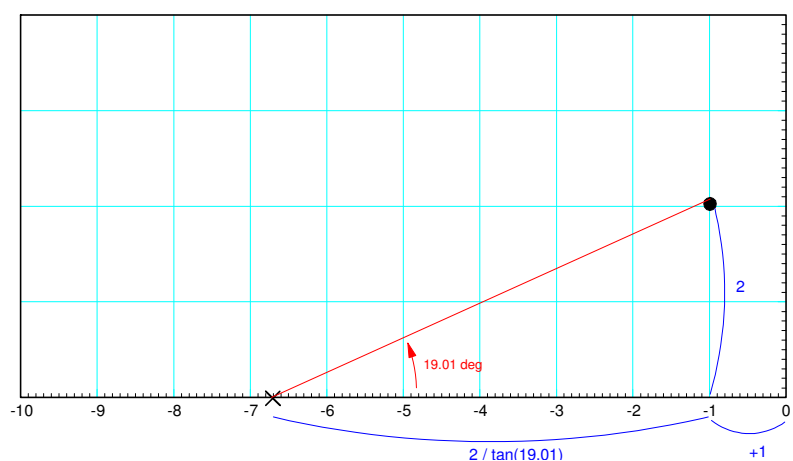This is 19.01 degrees short of 180 degrees, meaning the term we left out (1/s+a) must contribute that much phase

$$\angle\left(\frac{1}{s+a}\right)_{s=-1+j2} = -19.01^0$$

$$\angle(s+a) = 19.01^0$$

Using some trigonometry

$$a = \frac{2}{\tan\left(19.01^0\right)} + 1$$

$$a = 6.8046$$

Pick the pole at 'a' so that the angle from the pole to -1 + j2 is 19.01 degrees

This results in

$$K(s) = k\left(\frac{(s+0.3234)(s+2.081)}{s(s+6.8046)}\right)$$

Step 4) Pick 'k' so that GK = -1 at s = -1 + j2

With this K(s)

$$GK = \left(\frac{361.2378k}{s(s+5.439)(s+6.8046)(s+10.1)(s+15.65)}\right)$$

Evaluating at s = -1 + j2

$$\left(\frac{361.2378}{s(s+5.439)(s+6.8046)(s+10.1)(s+15.65)}\right)_{s=-1+j2} = 0.0392\angle180^0$$

The 180 degrees is a check that -1 + j2 is on the root locus.

The gain is off, however (it should be 1.00), so add a gain to make it one

$$k = \frac{1}{0.0392} = 25.49$$

resulting in

$$K(s) = 25.49\left(\frac{(s+0.3234)(s+2.081)}{s(s+6.8046)}\right)$$

Check in Matlab:

```
G = zpk([],[-0.3234, -2.081, -5.439, -10.1, -15.65], 361.2378 )

                361.2378
-----------------------------------------------
(s+0.3234) (s+2.081) (s+5.439) (s+10.1) (s+15.65)

K = zpk([-0.3234, -2.081],[0, -6.8046], 25.49 )

25.49 (s+0.3234) (s+2.081)
-------------------------
      s (s+6.805)

Gcl = minreal( G*K / (1 + G*K) )
eig(Gcl)

 -1.0000 + 2.0000i              pole we placed
 -1.0000 - 2.0000i
 -9.7606 + 4.0657i
 -9.7606 - 4.0657i
-16.4724
```
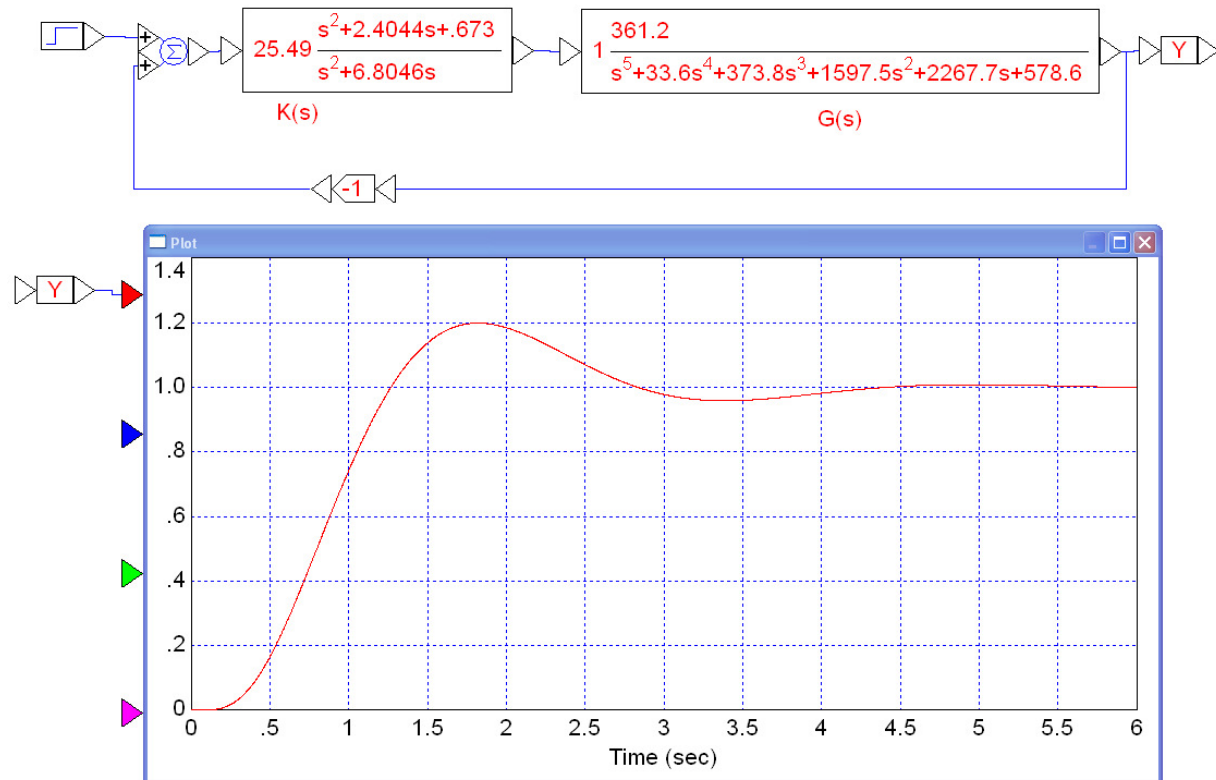
Note that the dominant pole is at -1 + j2, telling us that we designed K(s) correctly. The step response is then

```
t = [0:0.01:5]';
y = step(Gcl, t);
plot(t,y);
grid on
```
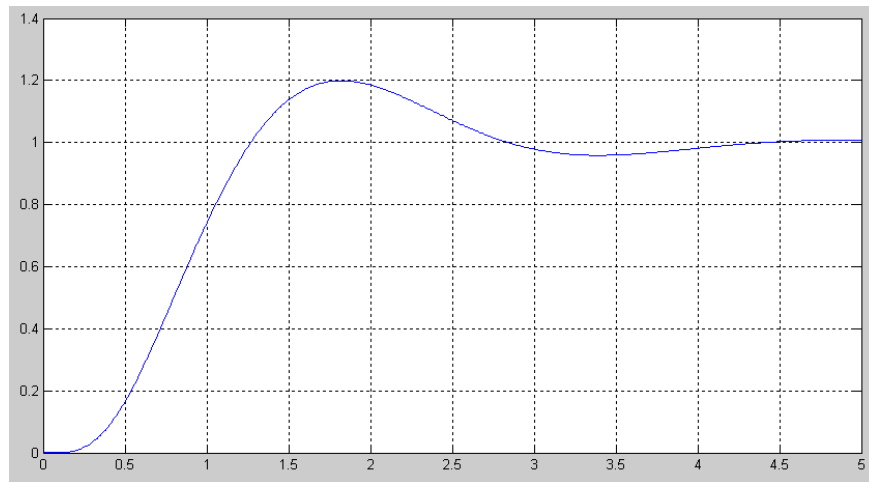
Step response of the closed-loop system

Checking in VisSim

Note that the requirements are met

- The DC gain is one
- The settling time is 4 seconds
- The overshoot is 20%, and
- K(s) has as many poles as zeros

Circuit Implementation

Recall

$$K(s) = 25.49 \left(\frac{(s+0.3234)(s+2.081)}{s(s+6.8046)} \right)$$

This is in a form we're not familiar with. If you rewrite it as

$$K(s) = \left(\frac{(s+0.3234)}{s} \right) \left(\frac{25.49(s+2.081)}{s+6.8046} \right)$$

then you can implement this as a PI compensator cascaded with a lead compensator. Note that multiplication is commutative. This means

- You can swap the zeros if you want, and
- You can distribute the gain of 25.49 between the two sections

In theory, it makes no difference how you split up the gains. In practice, try to keep the gain of each stage about the same to prevent saturating an op-amp.

Let

$$K(s) = \left(5.05 \frac{s+0.3234}{s} \right) \left(5.05 \frac{s+2.081}{s+6.8046} \right)$$
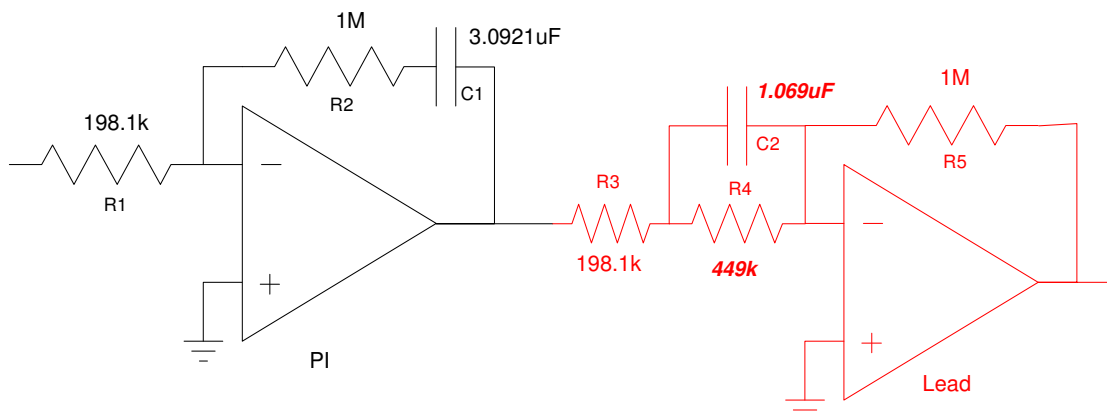
Circuit to implement $K(s) = \left(5.05 \frac{s+0.3234}{s} \right) \left(5.05 \frac{s+2.081}{s+6.8046} \right)$

Computations for the PI: Computations for the Lead:

$$K_1(s) = 5.05\left(\frac{s+0.3234}{s}\right)$$

Let R2 = 1M.   The high frequency gain is 5.05

$$\frac{R_2}{R_1} = 5.05$$

R1 = 198.1k. The zeros is at 0.3234

$$\left(\frac{1}{R_2 C}\right) = 0.3234$$

$$K(s) = 5.05\left(\frac{s+2.081}{s+6.8046}\right)$$

Let R5 = 1M. The high-frequency gain is 5.05

$$\frac{R_5}{R_3} = 5.05$$

R3 = 198.1k. The DC gain is 1.5444

$$\frac{R_5}{R_3+R_4} = 1.5444$$

R3 + R4 = 647.8k

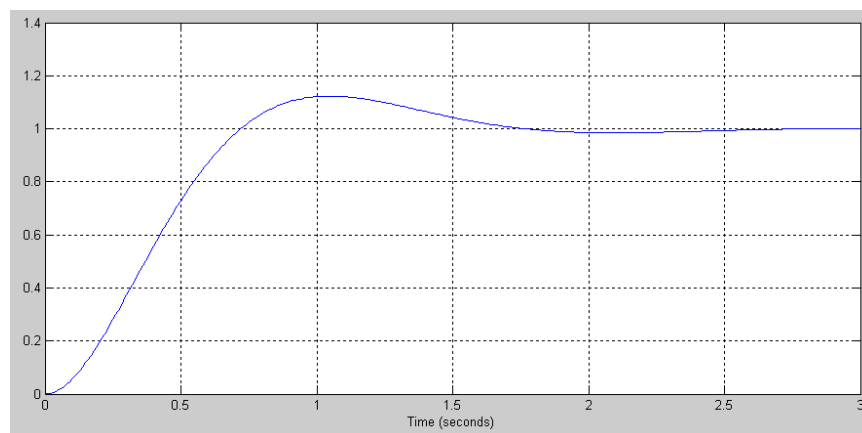R4 = 449.1k

The zero is at 2.081

$$\frac{1}{R_5 C_2} = 2.081$$

C2 = 1.069uF

Example 2:

Let

$$G(s) = \left(\frac{100}{(s+1)(s+3)(s+5)(s+10)}\right)$$

Design a compensator so that the closed-loop step response looks like

Desired Step Response

Solution:

Step 1: Translate the requirements to root-locus terms.

- The DC gain is one: the system should be type-1
- The 2% settling time is 2 seconds: the real part of the closed-loop dominant pole should be -2

- The overshoot is 12%: the damping ratio should be $\zeta = 0.55$, meaning the closed-loop dominant pole should be s = -2 + j3

Step 2: Find K(s) so that s = -2 + j3 is on the root locus. Start with

$$K(s) = \tfrac{1}{s}$$

Start canceling zeros until the phase of GK at s = -2 + j3 is below -180 degrees (or the root locus passes to the left of s = -2 + j3).

$$K(s) = \frac{(s+1)(s+3)(s+5)}{s}$$

$$GK = \left(\frac{100}{s(s+10)} \right)$$

Since you added three zeros, add three poles. One pole is at s = 0, making it a type-1 system. The other two are unknown. Say they're both at '-a'

$$K(s) = \left(\frac{(s+1)(s+3)(s+5)}{s(s+a)^2} \right)$$

$$GK = \left(\frac{100}{s(s+a)^2(s+10)} \right)$$

Find 'a' so that the angle of GK is 180 degrees at s = -2 + j3

$$\left(\frac{100}{s(s+10)} \right)_{s=-2+j3} = 3.2461 \angle -144.26^0$$

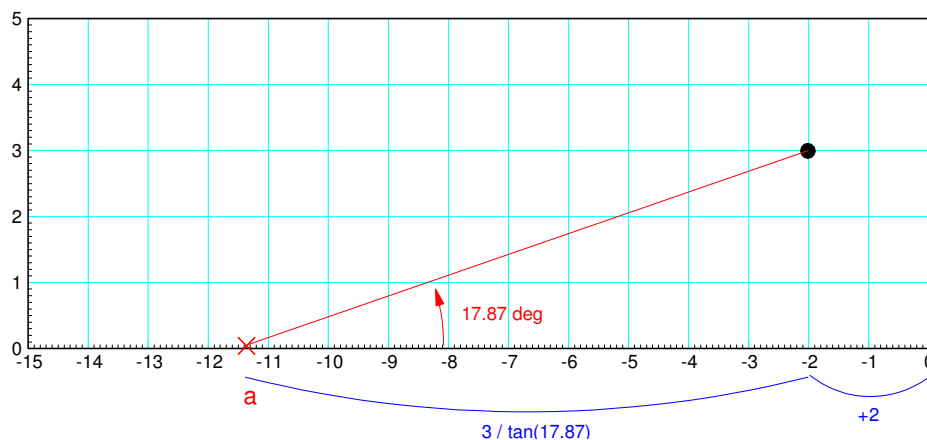To make the angles add up to 180 degrees, you need to subtract another 35.74 degrees

$$\angle \left(\frac{1}{(s+a)^2} \right)_{s=-2+j3} = -35.74^0$$

$$\angle(s+a)^2 = 35.74^0$$

$$\angle(s+a) = 17.87^0$$

Using some trigonometry

$$a = \frac{3}{\tan\left(17.87^0\right)} + 2 = 11.301$$

Two poles are placed at -1 so that the angle to s = -2 + j3 is 17.87 degrees

This results in

$$K(s) = \left(\frac{(s+1)(s+3)(s+5)}{s(s+11.301)^2} \right)$$

$$GK = \left(\frac{100k}{s(s+11.301)^2(s+10)} \right)$$

Step 3: Find k so that GK = -1 at s = -2 + j3

$$\left(\frac{100}{s(s+11.301)^2(s+10)} \right)_{s=-2+j3} = 0.0340\angle180^0$$

$$k = \frac{1}{0.0340} = 29.4222$$

and

$$K(s) = 29.4222 \left(\frac{(s+1)(s+3)(s+5)}{s(s+11.301)^2} \right)$$

Checking the step response in Matlab:

```
Gd = zpk([],[-2+j*3,-2-j*3],13);

G = zpk([],[-1,-3,-5,-10],100);
K = zpk([-1,-3,-5],[0,-11.301,-11.301],29.4222);
Gcl = minreal(G*K / (1+G*K));

eig(Gcl)

-14.3010 + 4.6697i
-14.3010 - 4.6697i
 -2.0000 + 3.0000i
 -2.0000 - 3.0000i

t = [0:0.01:3]';
yd = step(Gd, t);
```
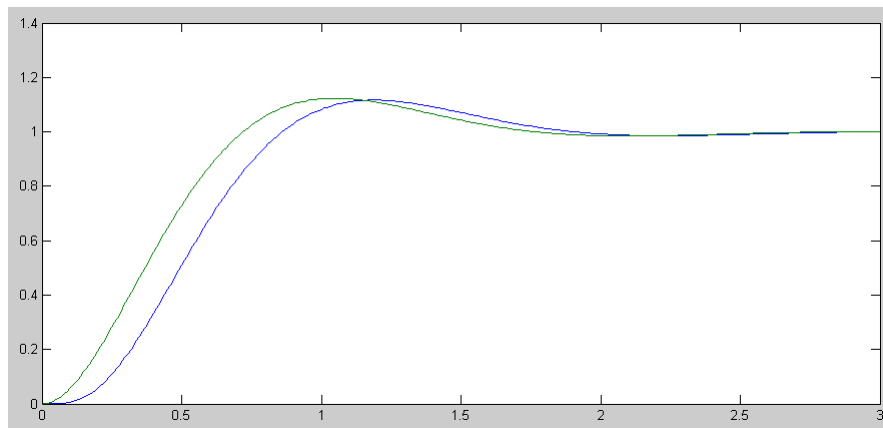
```
y = step(Gcl,t);
plot(t,y,t,yd)
```


Desired Step Response (green) and Actual Step Response (blue)

To implement K(s), split this into three first-order circuits

$$K(s) = 29.4222 \left(\frac{(s+1)(s+3)(s+5)}{s(s+11.301)^2} \right)$$

$$K(s) = \left(3.087 \left(\frac{s+1}{s} \right) \right) \left(3.087 \left(\frac{s+3}{s+11.301} \right) \right) \left(3.087 \left(\frac{s+5}{s+11.301} \right) \right)$$

This would then be a PI * Lead * Lead compensator