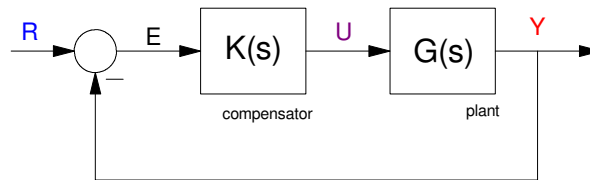


Lead Compensators using Root Locus

Introduction

Given a feedback system around a plant, $G(s)$



you want to choose K as large as possible since this results in

- Good tracking (error constants are proportional to k) and
- A fast response

The problem is that too much gain results in too much overshoot. Likewise, you usually pick the highest gain that meets your requirements, such as the overshoot being 20% or less.

If the resulting closed-loop system is too slow, you need to pull the root locus left somehow. That's the purpose of a lead compensator.

Lead Compensator Design (take 1)

To design a lead compensator, the first trick is to determine which pole to cancel. To see which one, consider the 5th-order system from our previous lecture:

$$G(s) = \left(\frac{361.2378}{(s+15.65)(s+10.1)(s+5.439)(s+2.081)(s+0.3234)} \right)$$

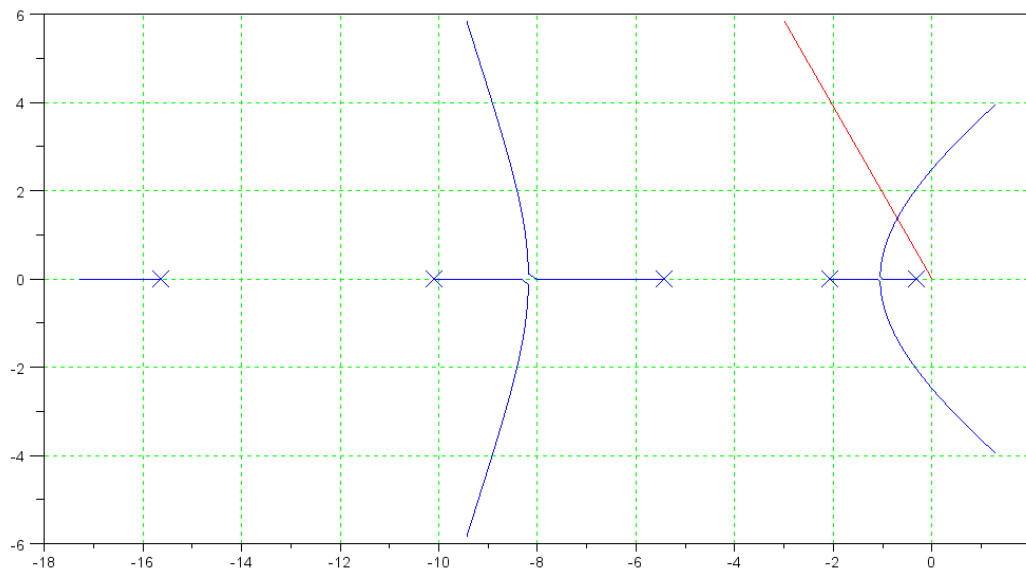
with the following root locus shown on the following page.

From before, with gain compensation ($K(s) = k$), the best you can do (meaning largest k) with the overshoot being no more than 20% is

Gain Compensation $K(s) = k$			
$K(s)$	Closed-Loop Dominant Pole(s)	K_p	2% Settling Time
5.5117	$s = -0.6942 + j1.3884$	3.4412	5.76 seconds

The dominant pole(s) are the right-most portion of the root locus. This is the part we want to shift left to speed up the system.

Clearly, canceling the fast pole at -15.65 and moving it left won't have much effect on the right-most portion of the root locus. That's not the pole we want to cancel. Instead, we want to cancel either the pole at -0.3234 or the pole at -2.081.



Root Locus for $G(s)$. $s = -0.7 + j1.36$ for a damping ratio of 0.4559

The dominant pole(s) are the right-most portion of the root locus. This is the part we want to shift left to speed up the system.

Clearly, canceling the fast pole at -15.65 and moving it left won't have much effect on the right-most portion of the root locus. That's not the pole we want to cancel. Instead, we want to cancel either the pole at -0.3234 or the pole at -2.081. To see which one you should cancel, design a lead compensator for each case.

If we cancel the pole at -0.3242, then

$$K(s) = k \left(\frac{s+0.3242}{s+3.242} \right)$$

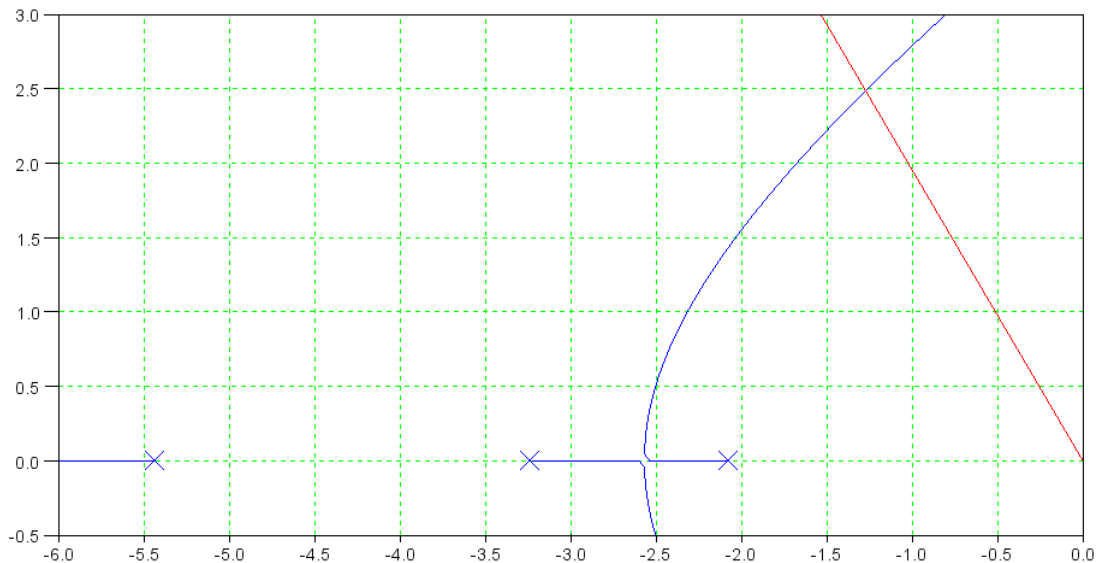
and

$$GK = \left(\frac{361.2378k}{(s+15.65)(s+10.1)(s+5.439)(s+2.081)(s+3.234)} \right)$$

Now, the problem of finding 'k' is the same as we did before:

- Find the point on the damping line which crosses the root locus.
- At this point, pick k so that $GK = -1$

Sketching the root locus of GK gives:



$$\text{Root Locus of } \left(\frac{361.2378k}{(s+15.65)(s+10.1)(s+5.439)(s+2.081)(s+3.234)} \right)$$

The point on the root locus that intersects the damping line is

$$s = -1.2531 + j2.5062$$

At this point

$$\left(\frac{361.2378}{(s+15.65)(s+10.1)(s+5.439)(s+2.081)(s+3.234)} \right)_{s=-1.2531+j2.5062} = 0.0654 \angle 180^\circ$$

As a check, the angle is 180 degrees, meaning that the point found is on the root locus. k is then

$$k = \frac{1}{0.0654} = 15.30$$

and

$$K(s) = 15.30 \left(\frac{s+0.3242}{s+3.242} \right)$$

This results in the 2% settling time being

$$T_{2\%} = \frac{4}{1.2531} = 3.1921 \text{ seconds}$$

The error constant, K_p , is

$$K_p = (G(s) \cdot K(s))_{s \rightarrow 0} = 0.9554$$

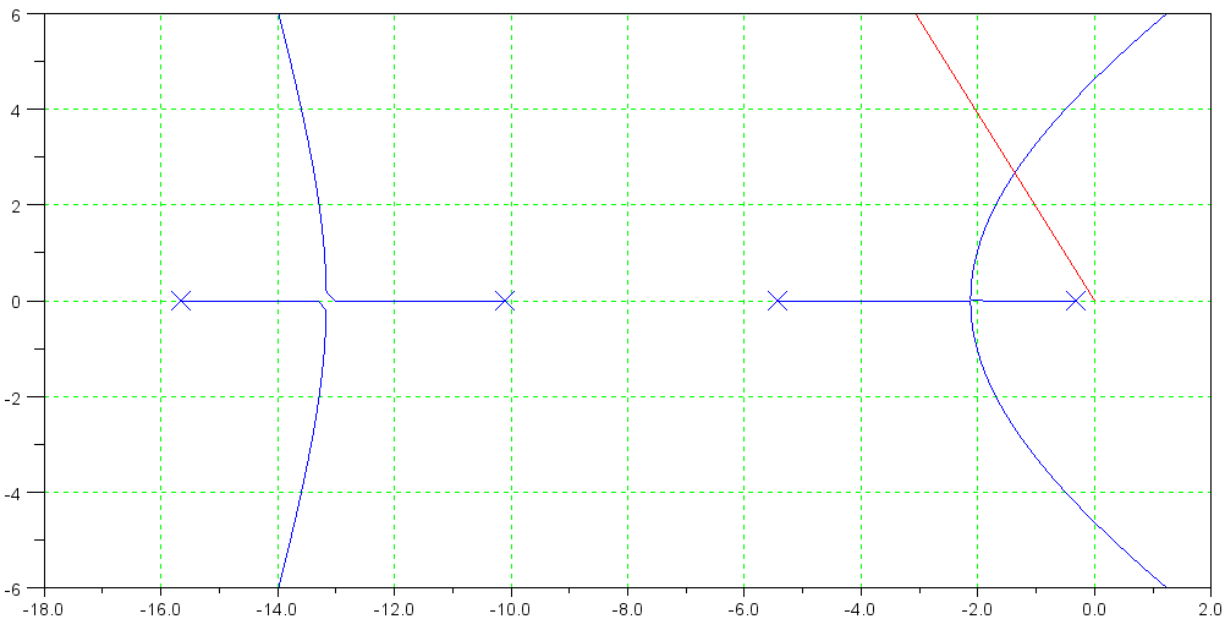
For comparison, cancel the next slowest pole at -2.081

$$K(s) = k \left(\frac{s+2.081}{s+20.81} \right)$$

This results in the open-loop system being

$$GK = \left(\frac{361.2378k}{(s+20.81)(s+15.65)(s+10.1)(s+5.439)(s+0.3234)} \right)$$

which results in the following (improved) root locus



Root Locus for $K(s) = k \left(\frac{s+2.081}{s+20.81} \right)$

For a damping ratio of 0.4559, find the spot on the (new) root locus which intersects this damping line

$$s = -1.3501 + j2.7002$$

At this point, pick k so that $GK = -1$

$$\left(\frac{361.2378}{(s+20.81)(s+15.65)(s+10.1)(s+5.439)(s+0.3234)} \right)_{s=-1.3501+j2.7002} = 0.0097 \angle 180^\circ$$

meaning

$$k = \frac{1}{0.0097} = 102.59$$

and

$$K(s) = 102.59 \left(\frac{s+2.081}{s+20.81} \right)$$

The resulting system will have a 2% settling time of

$$T_{2\%} = \frac{4}{1.3501} = 2.9627 \text{ seconds}$$

and an error constant of

$$K_p = (G(s) \cdot K(s))_{s=0} = 6.4052$$

These results are summarized in the following table:

Lead Compensation $K(s) = k (s+a) / (s+10a)$				
K(s)	Closed-Loop Dominant Pole(s)	U at t=0 K(s) as s → infinity	Kp	2% Settling Time seconds
5.5117	$s = -0.6942 + j1.3884$	5.5117	3.4412	5.76
$15.30 \left(\frac{s+0.3242}{s+3.242} \right)$	$s = -1.2531 + j2.5062$	15.30	0.9554	3.19
$102.59 \left(\frac{s+2.081}{s+20.81} \right)$	$s = -1.3501 + j2.7002$	102.59	6.4052	2.96

Note that canceling the pole at -2.081 resulted in

- A slightly faster system
- With a much larger error constant, Kp, meaning better tracking
- As well as a much larger value of U at t=0

This is due to the pole at -0.3242 being close to the origin. Ideally, we'd like it to be a type-1 system. It's not, but the pole at -0.3242 is closed to s=0, which improves the tracking at DC. You likewise want to keep this pole for tracking and cancel the next slowest pole: the one at -2.081

This results in the following rule of thumb:

Keep one pole at or near s=0.

Pick the zero of the lead compensator to cancel the next slowest (stable) pole.

The large value of U at t=0 results from the 10:1 ratio of the pole to the zero in the lead compensator - which makes the input at t=0 roughly 10x the input without the lead compensator. This larger input speeds up the system. It also places greater demands on the input and the system.

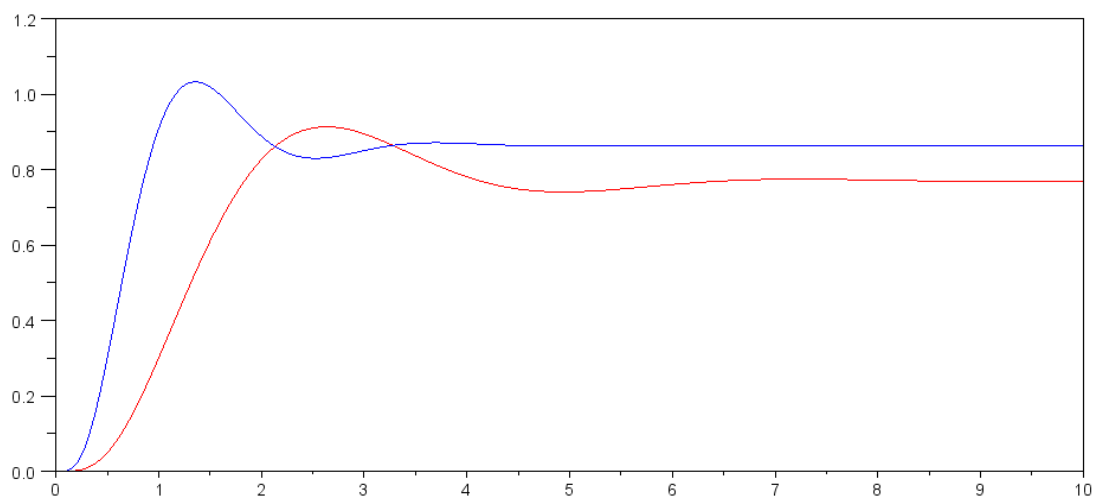
This results in the other rule of thumb for designing lead compensators:

Pick the pole 3 to 10 times larger than the zero

10 is a nice number in that it pulls the root locus further left, speeding up the system. It results in an input 10x larger than before, however, which can strain the system.

The step response of the lead compensated system is from the following Matlab code

```
>> G = zpk([], [-0.3243, -2.081, -5.439, -10.1, -15.65], 361.2378);  
>> K = zpk(-2.081, -20.81, 101.61)  
  
101.61 (s+2.081)  
-----  
      (s+20.81)  
  
>> t = [0:0.01:5]';  
>> Gcl = minreal(G*K / (1+G*K));  
>> y = step(Gcl, t);  
>> plot(t, y);
```



Step Response for the Gain Compensated System (red) and Lead Compensated System (blue).

Note that the lead compensator

- Sped up the system, and
- Reduced the steady-state error

3 is a nice number in that it only requires you to slam the input 3x harder to speed up the system, but it doesn't pull the root locus as far left.

Circuit Implementation of a Lead Compensator

To implement

$$K(s) = 101.61 \left(\frac{s+2.081}{s+20.81} \right)$$

use the following circuit. Note that it has 4 degrees of freedom (the R's and C's) with 3 constraints (the gains for K(s)). This means something is arbitrary. Let $R_2 = 1M$.

As $s \rightarrow \infty$,

- The capacitor shorts out R_b , resulting in one unknown (R_a)
- $K(s) = 101.61$

This tells you that

$$101.61 = \frac{R_2}{R_a}$$

$$R_a = 9.84k$$

At $s = 0$

- $K(s) = 10.161$
- The capacitor is an open circuit, meaning

$$\frac{R_2}{R_a + R_b} = 10.161$$

$$R_b = 88.56k$$

You also know this from the 10:1 ratio of the pole to the zero.

$$(R_a + R_b) = 10 R_a,$$

$$R_b = 9 R_a$$

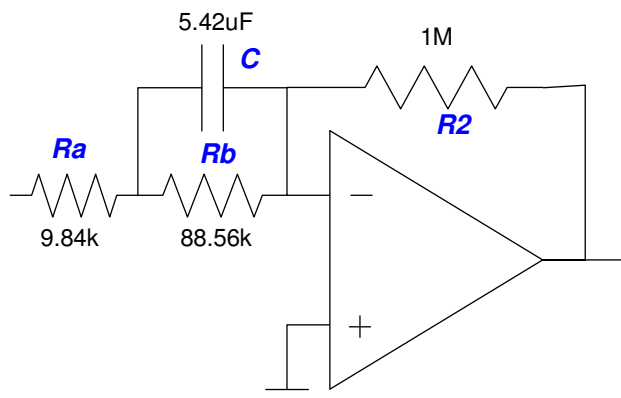
C makes R_a and R_b a variable resistor

- At low frequencies, the input resistance is $R_a + R_b$
- At high frequencies, the resistance is R_a
- You start to short out R_b when $R_b = 1 / C s$
- You start to short out R_b at 2.081 rad/sec

$$\frac{1}{R_b C} = 2.081$$

$$C = 5.42\mu F$$

The resulting op-amp circuit is thus:



Lead Compensator Circuit for K(s)

Software Implementation of a Lead Compensator

In software, this is implemented by making K(s) a proper fraction:

$$K(s) = 101.61 \left(\frac{s+2.081}{s+20.81} \right)$$

$$K(s) = 101.61 \left(\frac{s+20.81-18.729}{s+20.81} \right)$$

$$K(s) = 101.61 \left(1 - \frac{18.729}{s+20.81} \right)$$

Add a dummy state, Z, which is

```
Z = 0;
```

```
R = 100;
```

```
while(t < 100)
```

```
    E = R - V(10);
```

```
    dZ = -20.81*Z + 18.729*E;
```

```
    V0 = 101.61 * ( E - Z );
```

```
    dV(1) = 10*V0 - 20.1*V(1) + 10*V(2);
```

```
    dV(2) = 10*V(1) - 20.1*V(2) + 10*V(3);
```

```
    dV(3) = 10*V(2) - 20.1*V(3) + 10*V(4);
```

```
    dV(4) = 10*V(3) - 20.1*V(4) + 10*V(5);
```

```
    dV(5) = 10*V(4) - 20.1*V(5) + 10*V(6);
```

```
    dV(6) = 10*V(5) - 20.1*V(6) + 10*V(7);
```

```
    dV(7) = 10*V(6) - 20.1*V(7) + 10*V(8);
```

```
    dV(8) = 10*V(7) - 20.1*V(8) + 10*V(9);
```

```
    dV(9) = 10*V(8) - 20.1*V(9) + 10*V(10);
```

```
    dV(10) = 10*V(9) - 10.1*V(10);
```

```
    Z = Z + dZ*dt;
```

```
    V = V + dV*dt;
```



```
t = t + dt;
.
.
.
```

Lead Compensators (take 2).

If the system still isn't fast enough, add another lead compensator. Pick the second lead compensator to cancel the next slowest pole.

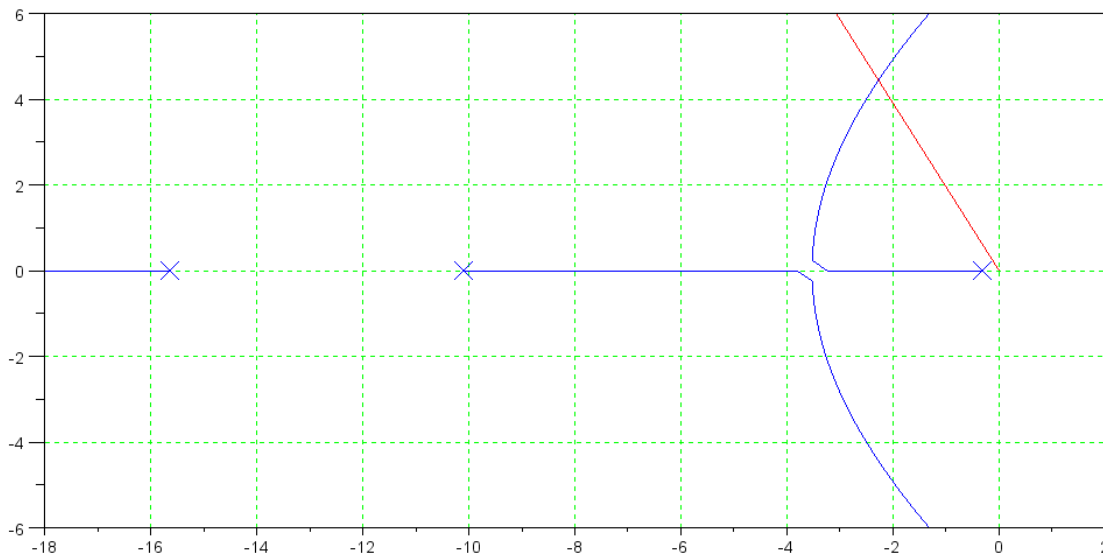
$$K(s) = k \left(\frac{s+2.081}{s+20.81} \right) \left(\frac{s+5.439}{s+53.39} \right)$$

$$GK = \left(\frac{361.2378}{(s+15.65)(s+10.1)(s+5.439)(s+2.081)(s+0.3234)} \right) \cdot k \left(\frac{s+2.081}{s+20.81} \right) \left(\frac{s+5.439}{s+53.39} \right)$$

$$GK = \left(\frac{361.2378k}{(s+15.65)(s+10.1)(s+54.39)(s+20.81)(s+0.3234)} \right)$$

Now the root locus looks like the following:

```
>> k = logspace(-2, 2, 1000)';
>> GK = zpk([], [-15.65, -10.1, -5.439, -20.81, -3.234], 361.2378);
>> rlocus(GK, k);
```



Root Locus for G(s) with two lead compensators

The point on this root locus which intersects the 0.4559 damping line is

$$s = -2.2463 + j4.4925$$

At this point

$$\left(\frac{361.2378k}{(s+15.65)(s+10.1)(s+54.39)(s+20.81)(s+0.3234)} \right)_{s=-2.2463+j4.4925} = 0.0006 \angle 180^\circ$$

resulting in

$$k = \frac{1}{0.006} = 1729.60$$

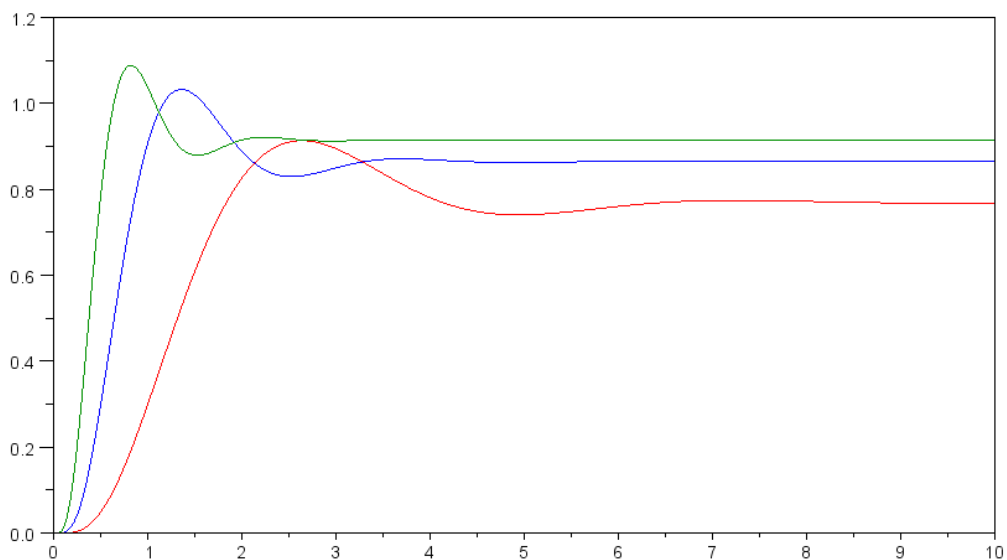
meaning

$$K(s) = 1729.60 \left(\frac{s+2.081}{s+20.81} \right) \left(\frac{s+5.439}{s+53.39} \right)$$

With this second lead compensator, the system's specifications are:

Lead Compensation $K(s) = k (s+a) / (s+10a)$				
$K(s)$	Closed-Loop Dominant Pole(s)	U at t = 0 K(s) as s -> infinity	Kp	T _{2%} seconds
5.5117	s = -0.6942 + j1.3884	5.5117	3.4412	5.76
$15.30 \left(\frac{s+0.3242}{s+3.242} \right)$	s = -1.2531 + j2.5062	15.3	0.9554	3.19
$102.59 \left(\frac{s+2.081}{s+20.81} \right)$	s = -1.3501 + j2.7002	102.59	6.4052	2.96
$1729.60 \left(\frac{s+2.081}{s+20.81} \right) \left(\frac{s+5.439}{s+53.39} \right)$	s = -2.2463 + j4.4925	1729.60	10.7987	1.78

The closed-loop system's step response is as follows:



Step Response for the Gain (red), Lead (blue) and Two Lead (green) Compensated Systems

Note, as expected, the 2nd lead compensator

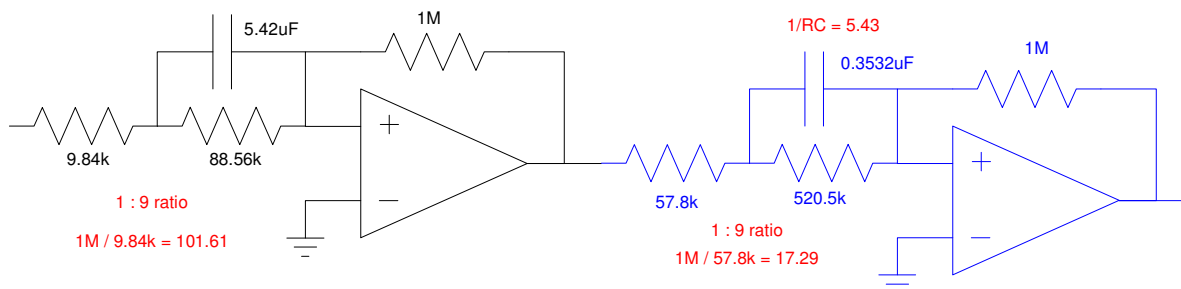
- Speeds up the closed-loop system,
- Gives better tracking, and
- Requires the input at $t=0$ to be 313x larger than it was with a gain compensator.

In theory, you can make a Chevette go from 0 to 60mph in 0.1 second. In practice, the input is limited. Once you start to saturate the input, there's no point in trying to speed up the system.

A circuit to implement this compensator would be

$$K(s) = 1729.60 \left(\frac{s+2.081}{s+20.81} \right) \left(\frac{s+5.439}{s+54.39} \right)$$

$$K(s) = \left(101.61 \frac{s+2.081}{s+20.81} \right) \left(17.29 \frac{s+5.439}{s+54.39} \right)$$



Circuit to implement a 2-stage lead compensator