

Root Locus with Repeated and Complex Poles & Zeros

Root Locus with Multiple Poles:

From the lecture on error constants

- One pole at $s=0$ is good: it causes the system to have no error for a step input
- Two poles at $s = 0$ is better: it causes the system to have no error for a step or ramp input

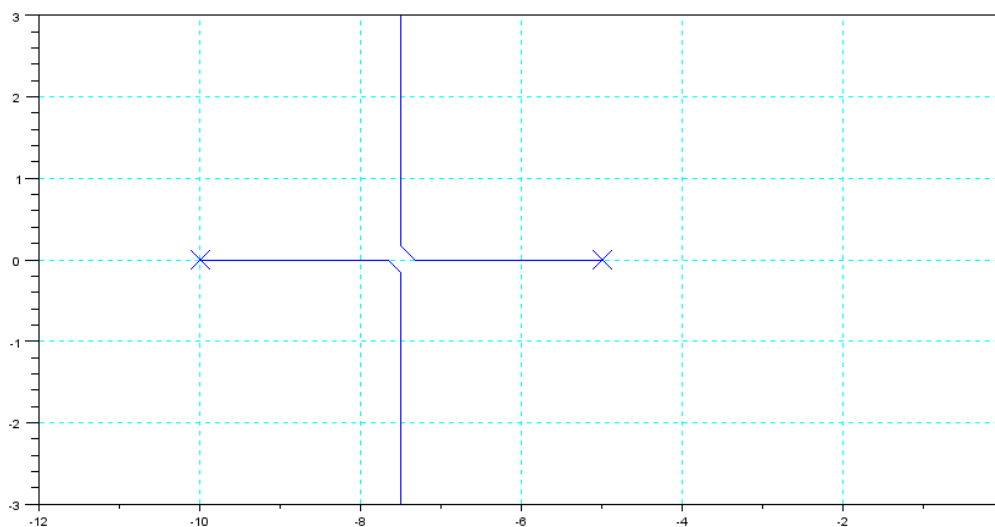
So, why not add a bunch of poles at $s = 0$?

The problem with this can be seen on a root-locus plot. For example, compare the root locus plots for

- $G(s) = \left(\frac{10}{(s+5)(s+10)} \right)$ *a type-0 system*
- $G(s) = \left(\frac{10}{s(s+5)(s+10)} \right)$ *a type-1 system*
- $G(s) = \left(\frac{10}{s^2(s+5)(s+10)} \right)$ *a type-2 system*
- $G(s) = \left(\frac{10}{s^3(s+5)(s+10)} \right)$ *a type-3 system*

Type-0 System: For this system, stabilizing the system is fairly easy: the root locus starts in the left half plane and remains there.

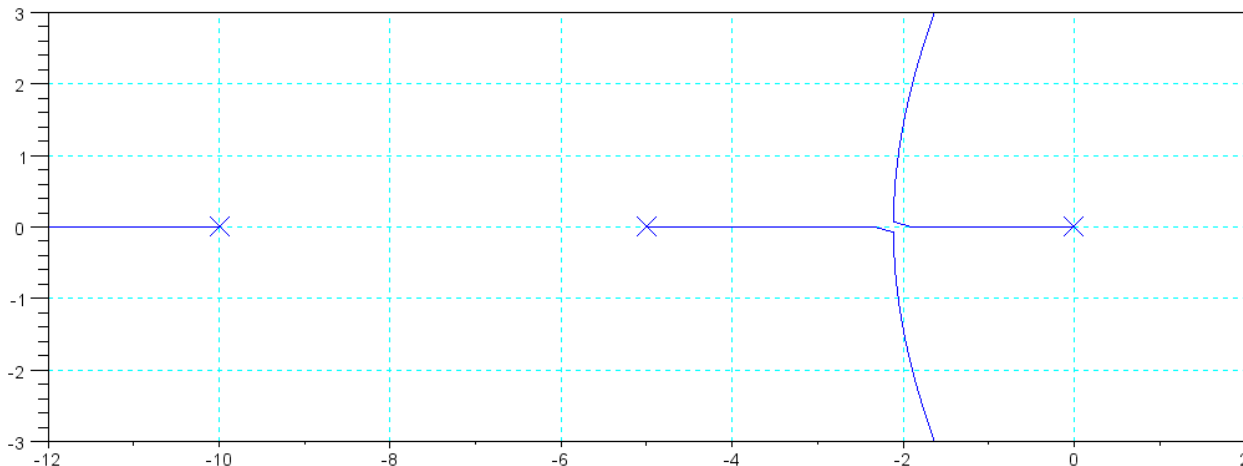
```
>> G0 = zpk([], [-5, -10], 10)
>> k = logspace(-2, 2, 1000)';
>> rlocus(G0, k);
```



Root Locus for a Type-0 System

Type 1 System: Again, stability isn't a problem. The pole at $s=0$ moves left, resulting in a stable system. If you make the gain too large, it goes unstable again, but for small gains it should work.

```
>> G1 = zpk([], [0, -5, -10], 10)
>> k = logspace(-2, 2, 1000)';
>> rlocus(G0, k);
```

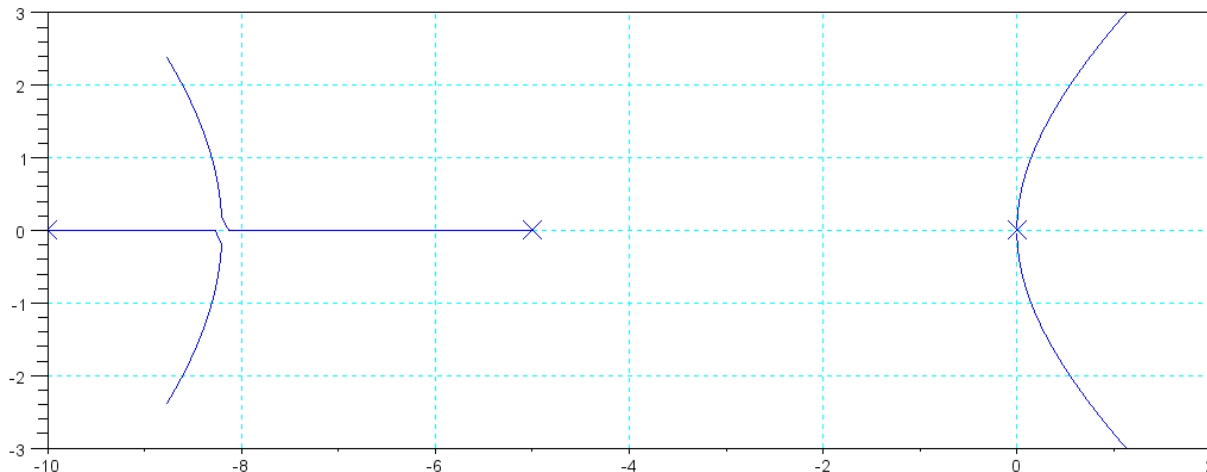


Root Locus for a Type-1 System

Type 2 System: Now you have a problem. The two poles at $s = 0$ split apart at ± 90 degrees (think of it as having poles at $\{ +0.001, -0.001 \}$). These come together at $s = 0$ and shoot up and down.) This locus is then pushed to the right by the other poles, making the system unstable.

You can stabilize a type-2 system, but

- You will need zeros to attract the root locus left,
- Even then, the system will tend to be very oscillatory.



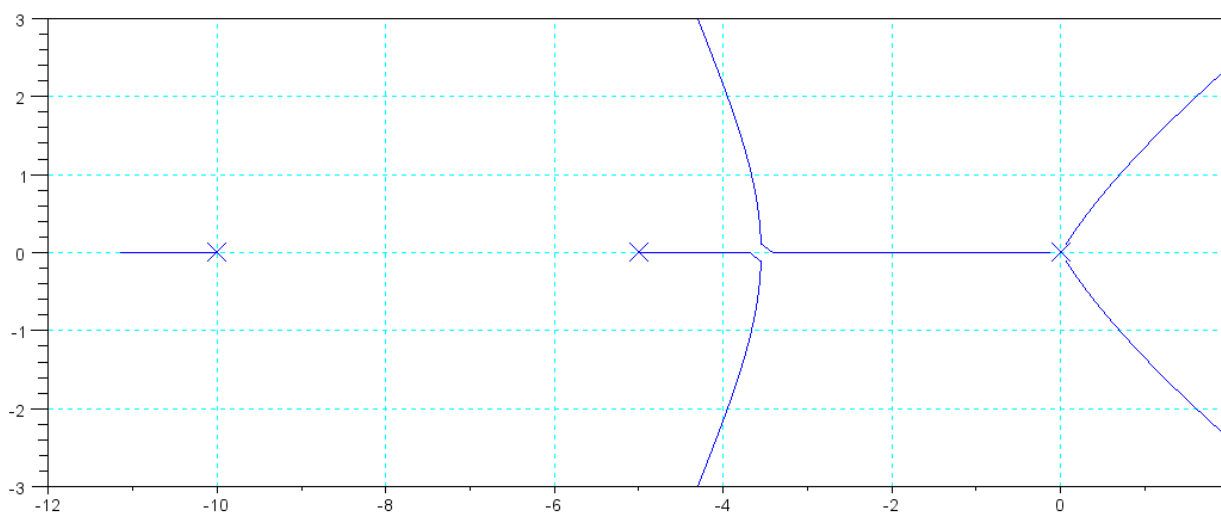
Root Locus for a Type-2 System: The poles at $s = 0$ become oscillatory and unstable

Matlab Code:

```
>> G2 = zpk([], [0, 0, -5, -10], 10)
>> k = logspace(-2, 2, 1000)';
>> rlocus(G0, k);
```

Type 3 Systems: Problems get worse. The three poles at $s = 0$ split apart with two going right at an angle of ± 60 degrees (think of it as having poles at $\{ +0.001, 0, -0.001 \}$). The poles at $+0.001$ and 0 come together then get pushed right by the 3rd pole.)

It is very difficult to pull all roots into the left half plane. It is very difficult to stabilize a type-3 system.



Type-3 system: Two of the poles at $s = 0$ go right immediately, making the system unstable.

Moral:

- Don't try to make the system Type-2 or higher: it's going to cause problems.
- Usually, *do* strive for a type-1 system. Type-1 systems are fairly easy to stabilize and result in no error in a step input.

Complex Poles and Departure Angles

So far, we've looked at real poles. If you have a complex pole (and its conjugate), not much changes as far as real-axis loci, $j\omega$ crossing, etc. The angle the root locus leaves the complex pole (termed the departure angle) would help to know, however.

To compute the departure angle, note that any point on the root locus satisfies the condition

$$\text{angle}(GK) = 180^\circ$$

or

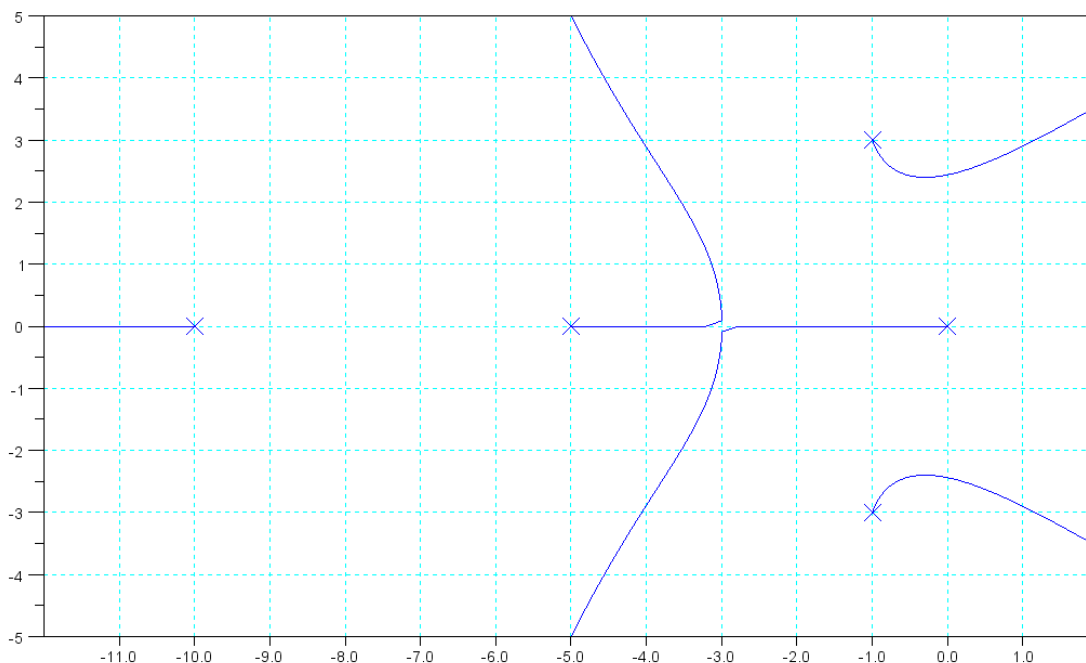
$$\sum (\text{angles from poles to } s) - \sum (\text{angles from zeros to } s) = 180^\circ$$

Near a complex pole, you can evaluate GK for all terms except the complex pole (since you will get zero). Close to the pole, however, the angles have to add up to 180 degrees. The departure angle is whatever the angle has to be so that the angles add up.

Example: For the root locus of $G(s)$:

$$G(s) = \left(\frac{1000}{s(s+1+j3)(s+1-j3)(s+5)(s+10)} \right)$$

Compute the departure angle from the pole at $s = -1 + j3$:



Root Locus of $G(s) = \left(\frac{1000}{s(s+1+j3)(s+1-j3)(s+5)(s+10)} \right)$. Compute the departure angle from the pole at $-1 + j3$

At any point on the root locus, the angles must add up to 180 degrees. Evaluate $G(s)$ at $s = -1 + j3$:

$$\left(\frac{1000}{s(s+1+j3)(s+1-j3)(s+5)(s+10)} \right)_{s=-1+j3}$$

This you can't do since one term goes to zero. So, evaluate everything else:

$$\left(\frac{1000}{s(s+1+j3)(s+5)(s+10)} \right)_{s=-1+j3} \cdot \left(\frac{1}{s+1-j3} \right)_{s=-1+j3} = \alpha \angle 180^\circ$$

$$(1.111 \angle 106.2^\circ) \cdot \left(\frac{1}{s+1-j3} \right)_{s=-1+j3} = \alpha \angle 180^\circ$$

For the angles to add up to 180 degrees

$$\text{angle} \left(\left(\frac{1}{s+1-j3} \right)_{s=-1+j3} \right) = 73.74^\circ$$

$$\text{angle} \left((s+1-j3)_{s=-1+j3} \right) = -73.74^\circ$$

The departure angle is -73.74 degrees. Note that this matches the angle that Matlab drew on the root-locus plot.

Complex Zeros and Approach Angles

Note that

- Poles repel the root locus plot.
- Zeros attract the root locus plot.

If you have a zero, it will try to pull the root locus to the zero.

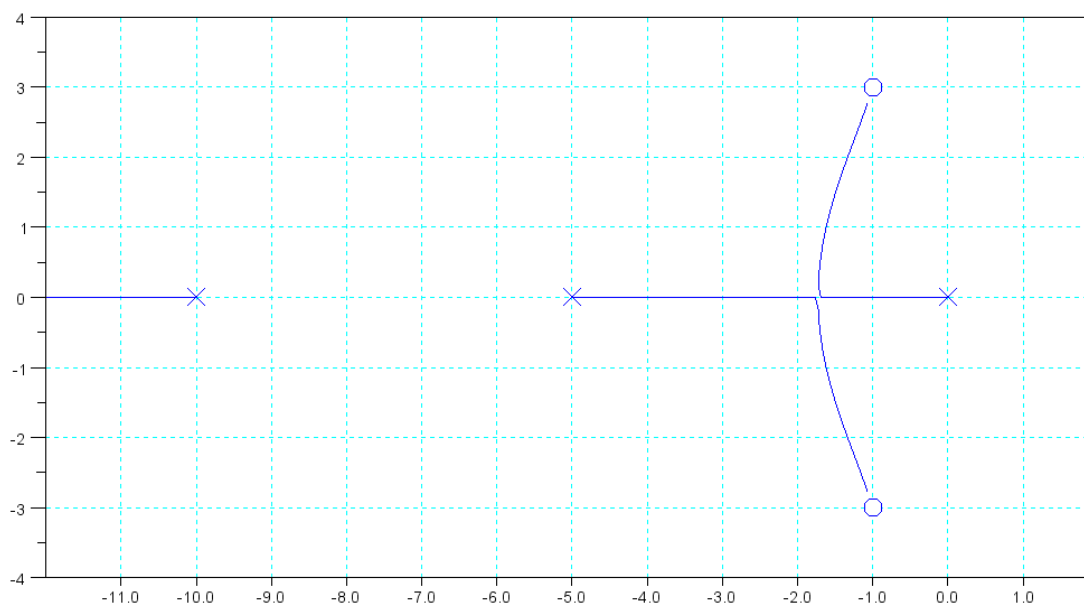
If you have a complex zero, it will attract one of the root locus plots. The angle the root locus goes into the complex zero is termed *the approach angle*.

Computing approach angles are very much like departure angles: it's whatever it takes for the angles to add up to 180 degrees.

Example: Compute the approach angle to the zero at $-1 + j3$:

$$G(s) = \left(\frac{(s+1+j3)(s+1-j3)}{s(s+5)(s+10)} \right)$$

The root-locus from Matlab is:



Root Locus for $\left(\frac{(s+1+j3)(s+1-j3)}{s(s+5)(s+10)}\right)$

To find the approach angle to the zero at $-1 + j3$, note that the angles must add up to 180 degrees:

$$\left(\frac{(s+1+j3)(s+1-j3)}{s(s+5)(s+10)}\right)_{s=-1+j3} = \alpha \angle 180^\circ$$

You can't evaluate the zero at $-1+j3$ since the amplitude is zero. Evaluating everything else results in

$$\left(\frac{(s+1+j3)}{s(s+5)(s+10)}\right)_{s=-1+j3} \cdot \left(\frac{s+1-j3}{1}\right)_{s=-1+j3} = \alpha \angle 180^\circ$$

$$(0.04 \angle -73.74^\circ) \cdot \left(\frac{s+1-j3}{1}\right)_{s=-1+j3} = \alpha \angle 180^\circ$$

For the angles to add up

$$\text{angle}\left(\left(\frac{s+1-j3}{1}\right)_{s=-1+j3}\right) = -106.26^\circ$$

The approach angle to the zero at $-1 + j3$ is -106.26 degrees