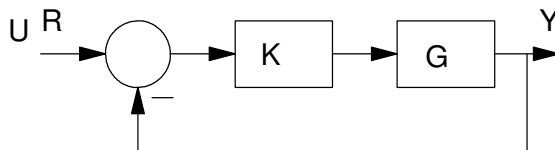


Root Locus

Background:

Routh Criteria tells you the range of gains that result in a stable system. It doesn't tell you how the system will behave, however. That's a problem.

For example, for the following system, find 'k' for a 'good' response, where 'good' means a fast response with less than 10% overshoot for step input.



Root Locus: Find the roots of the closed-loop system for $0 < k < \infty$

$$G(s) = \left(\frac{1000}{s(s+5)(s+20)} \right)$$

The open-loop system is

$$GK = \left(\frac{1000k}{s(s+5)(s+20)} \right)$$

while the closed-loop system is

$$\left(\frac{GK}{1+GK} \right) = \left(\frac{1000k}{s(s+5)(s+20)+1000k} \right)$$

Using a Routh table, the range of k for stability can be found to be

$$0 < k < 2.5$$

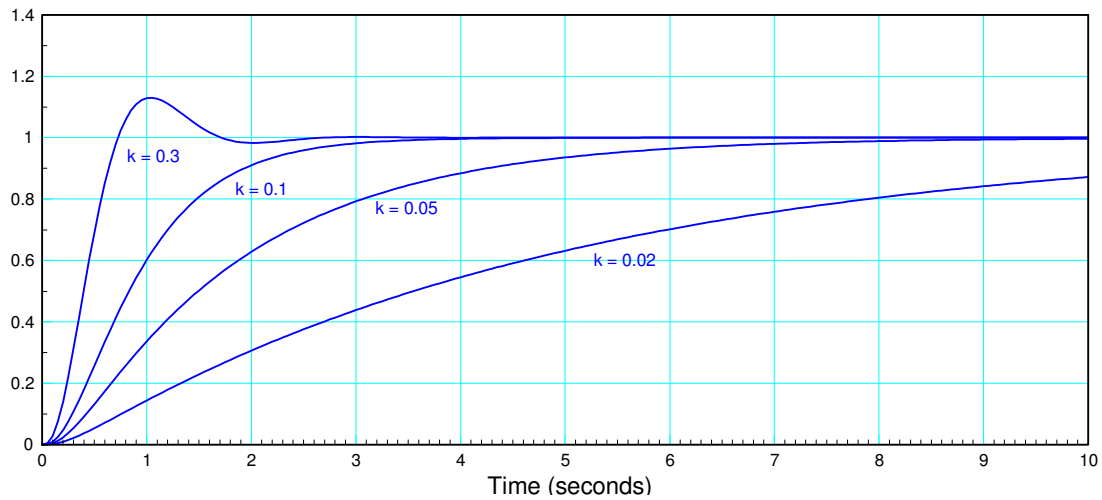
That doesn't really help. If you plot the step response for various values of k in this range, you get the following step responses:

Matlab Code:

```
t = [0:0.05:10]';
G = zpk([], [0, -5, -20], 1000)
k1 = 0.02;
k2 = 0.05;
k3 = 0.1;
k4 = 0.3;

y1 = step(G*k1 / (1+G*k1), t);
y2 = step(G*k2 / (1+G*k2), t);
y3 = step(G*k3 / (1+G*k3), t);
y4 = step(G*k4 / (1+G*k4), t);

plot(t, y1, t, y2, t, y3, t, y4)
```



Step Response for $k = \{ 0.02, 0.05, 0.1, 0.3 \}$

Note from the step responses:

- All four plots are stable as the Routh Criteria predicts: they are all in the range of $0 < k < 2.5$
- The response for each one is different - meaning that the closed-loop poles are shifting
- The response isn't random: there is a smooth progression from one step response to the next.

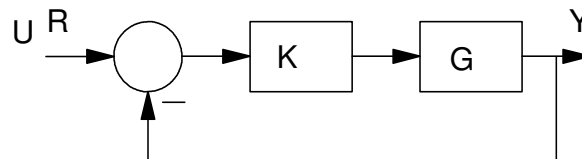
This is basically the concept of a root-locus plot.

- When you close the feedback loop, the poles shift.
- If you could plot how the poles are shifting, as k varies, you'll know what options you have for what the closed-loop system's behaviour.

This plot is called a *root locus plot* since it plots the roots of the closed-loop system.

Definitions:

Assume a feedback system of the following form:



Root Locus: Find the roots of the closed-loop system for $0 < k < \infty$

where

$$GK = \left(\frac{k \cdot z(s)}{p(s)} \right)$$

Open-Loop Gain:	GK
Closed-Loop Gain:	$\left(\frac{kG}{1+kG}\right)$
Open-Loop Poles:	solutions to $p(s) = 0$
Open-Loop Zeros:	solutions to $z(s) = 0$.
Closed-Loop Poles:	solutions to $p(s) + kz(s) = 0$.
Root Locus:	Plot of the closed-loop poles as k varies from 0 to infinity.

Sketching a Root Locus Plot (method #1)

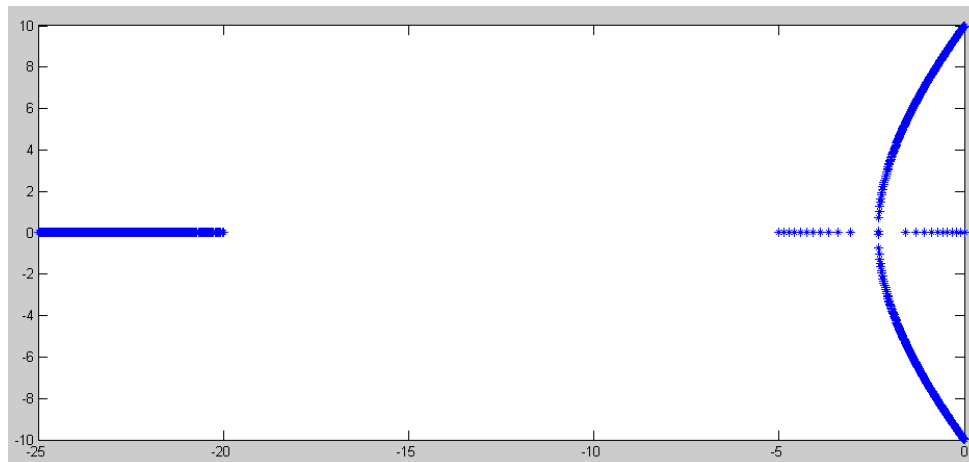
One way to sketch a root locus plot is to compute the closed-loop poles for a large number of gains and plot these on a graph. It isn't very efficient, but it's something MATLAB is good at.

Example: Sketch the roots of the following polynomial for $0 < k < 2.5$

$$s(s+5)(s+20) + 1000k = 0$$

Solution: In Matlab

```
p = poly([0, -5, -20])
      1      25      100      0
z = [0, 0, 0, 1000]
      0      0      0      1000
k = [0:0.01:2.5]';
for i=1:length(k)
    R = roots(p + k(i)*z);
    plot(real(R), imag(R), '*');
    hold on
end
```



Method 1 for Drawing a Root Locus: Find the roots for 2500 values of k

Note that this plot is consistent with what we saw on the previous step responses:

- The dominant pole starts out near $s = 0$ and moves left. As the gain increases, the pole at the origin speeds up.
- As k increases, the dominant pole becomes complex, resulting in overshoot in the step response.

Method 2: Matlab Function *rlocus*

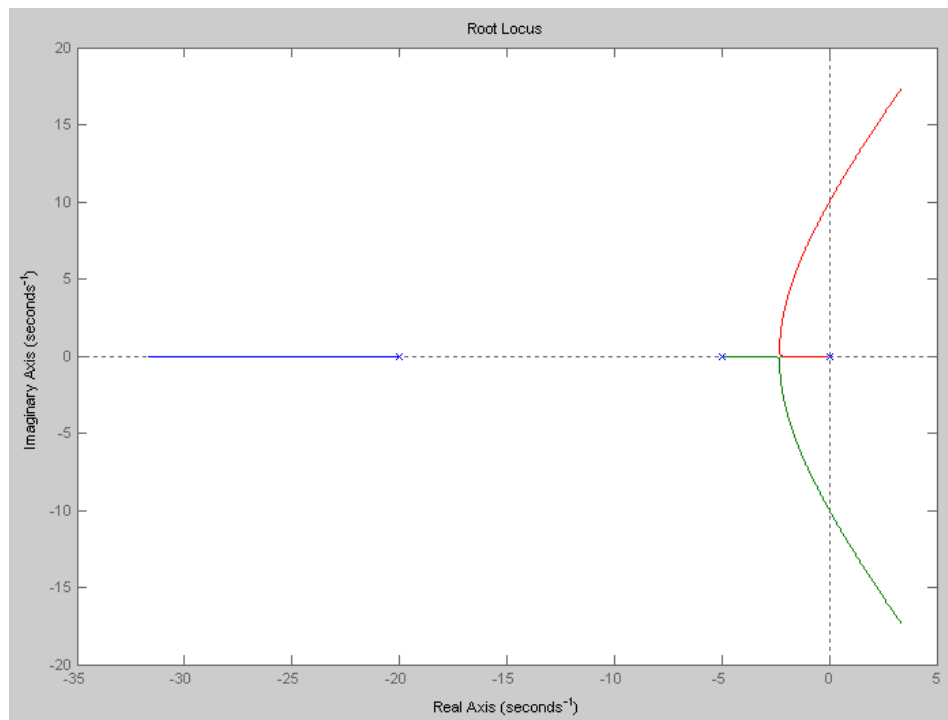
A second method uses the function *rlocus*

```
R = rlocus(G, k);  
  
R = closed-loop roots  
G = system, G(s)  
k = gains to evaluate
```

This is actually identical to method #1, just in an m-file

Example:

```
G = zpk([], [0, -5, -20], 1000);  
k = logspace(-2, 1, 1000)';  
rlocus(G, k);
```

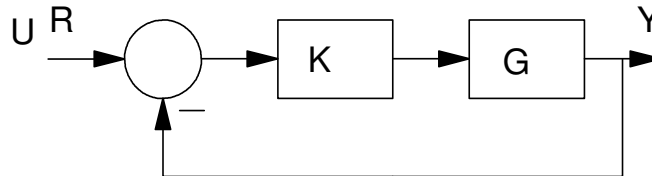


Root Locus Plot from the Matlab function *rlocus(G, k)*

Note that the Matlab function *rlocus* is convenient, but it doesn't give you much insight for how and why the root locus looks like this.

Method #3: Graphical Methods

Assume you have a unity feedback system as follows:



where

$$KG = \left(\frac{kz(s)}{p(s)} \right)$$

As you vary k from 0 to infinity, the dynamics of the closed-loop system will vary. The closed-loop system will be

$$\left(\frac{kG}{1+kG} \right) = \left(\frac{k \cdot z(s)}{p(s) + k \cdot z(s)} \right).$$

The roots of the closed-loop system determine how the system will behave. Hence, we are interested in the solutions to

$$p(s) + k \cdot z(s) = 0$$

A root locus plot is a plot of the solutions to $p(s)+kz(s)=0$ for $0 < k < \infty$.

Rewriting this

$$\frac{p(s)}{z(s)} = -k = k \angle 180^\circ$$

Since 's' can be complex, this graphical interpretation of $-k = \frac{p(s)}{z(s)}$ provides two pieces of information: one for amplitude and one for the angle:

1) The amplitude of both sides must match, meaning

$$k = \frac{\prod(\text{distances from the poles to point } s)}{\prod(\text{distances from the zeros to point } s)}$$

2) The angles must match:

$$180^\circ = \sum (\text{angles from the poles to point } s) - \sum (\text{angles from the zeros to point } s)$$

Plotting all points on the 's' plane which satisfies the angle condition produces the root-locus. The root locus tells you how the system can behave. Once you pick a point on the root locus, k is found using the first equation.

Rules for Plotting the Root Locus:

Recall that the root-locus plot is a plot of all points which satisfy

$$p(s) + k \cdot z(s) = 0$$

which is also a plot of all points which satisfy the angle condition. From (3), two solutions are obvious:

1. Poles:

At $k=0$, the roots start at the open-loop poles. (i.e. the solution to $p(s)=0$). Mark an 'X' at these points.

2. Zeros:

As $k \rightarrow \infty$, the roots go to the open-loop zeros (i.e. the solution to $z(s)=0$). Mark an 'O' at these points.

3. Real Axis Loci:

Any point on the real axis where the angles add to 180 degrees will be on the root locus. Note that

- If a real pole is to the left of point 's', the angle from the pole to this point is zero degrees.
- If a real pole is to the right of point 's', the angle from the pole to point 's' is 180 degrees.
- Complex poles have no effect. Two poles to the left add zero degrees (twice). Two poles to the right add $360^\circ = 0^\circ$.
- Zeros act like poles since $+180^\circ = -180^\circ$.
- Adding 180 an odd number of times produces 180.
- Adding 180 an even number of times produces 00.

Hence,

Real axis loci will be whenever there are an odd number of poles and zeros to the right.

4. Number of Asymptotes:

Assume there are m zeros and n poles. m of the poles will go to the m zeros. The remainder go to infinity.

The number of asymptotes = n-m. (# of poles - # of zeros)

5. Asymptote Angle:

Assume that there are m zeros and n poles. Then, k is from

$$-k = \frac{p(s)}{z(s)} = \frac{s^n + \dots}{s_m + \dots}$$

If $k \rightarrow \infty$ then $s \rightarrow \infty$ as well. In this case, only the highest power of 's' matters:

$$-k = \frac{s^n}{s^m} = s^{n-m}$$

The angle is then

$$180^\circ = (n - m)\angle\phi$$

where ϕ is the angle of 's'.

The asymptote angles are: $\phi = \frac{180^\circ + k \cdot 360^\circ}{n - m}$ (k=0, 1, 2, ...)

6. Asymptote Intersect:

At infinity, the poles and zeros look like they are at the same spot on the 's' plane (relatively). This will be the 'average' pole and zero location

$$\text{Asymptote Intersect} = \frac{\sum(\text{poles}) - \sum(\text{zeros})}{\#\text{poles} - \#\text{zeros}}$$

7. Breakaway Point:

If two real-axis poles come together, they will move away from the real axis at a point. Two ways to compute these points are

- Treat each pole as a - charge and each zero as a + charge. An electron at equilibrium on the real axis will be at a breakaway point.
- Solve $\frac{d}{ds} \left(\frac{p(s)}{z(s)} \right) = 0$. The sensitivity of the root locus at the breakaway point is infinity (a small change in k produces a large change in the root location.)

8. jw Crossings:

These are important since they tell you what gain makes the system go unstable:

$$\text{angle}(G(s))_{s=j\omega} = 180^\circ$$

9. Departure Angle:

If a pole or zero is complex, the departure angle is the angle at which the root locus leaves the pole or enters the zero. This is found by summing the angles to 180 degrees.

10. Gain at a point on the root-locus:

The root locus plots all possible locations that the system's roots can be for $0 < k < \infty$. The value of k at any particular point on the root locus comes from

$$\left(\frac{kz(s)}{p(s)} \right) = kG(s) = -1.$$

The minus sign means that the angles must add to 180 degrees. The amplitude tells you the value of k .

Root Locus Summary $p(s) + k z(s) = 0$	
Poles	solution to $p(s)=0$
Zeros	solution to $z(s)=0$
Real Axis Loci:	There are an off number of poles or zeros to the right
Breakaway Point(s)	$\frac{d}{ds} \left(\frac{p(s)}{z(s)} \right) = 0$
jw crossing	$\text{ang}(G(jw)) = 180^\circ$
Number of asymptotes	$n-m$ (#poles - #zeros)
Asymptote Angles	$\frac{180^\circ + k \cdot 360^\circ}{n-m}$
Asymptote Intersect	$\frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$
Gain at any point on the root-locus	$k \cdot G(s) = -1$

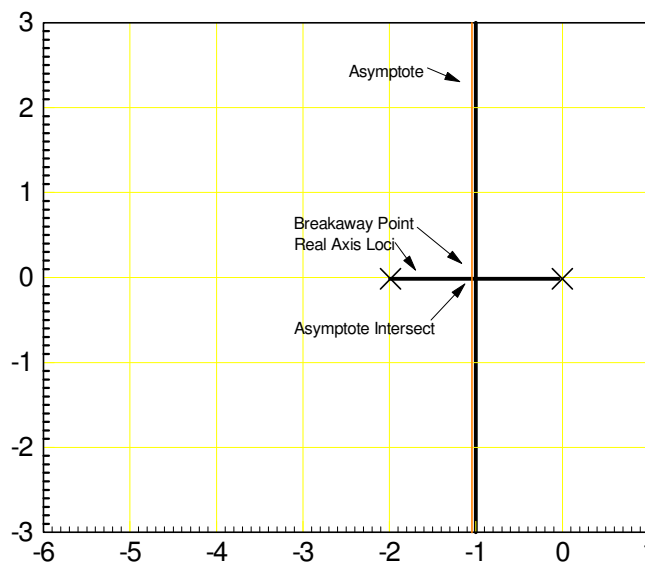
Problem 1a: Sketch the root locus of

$$G(s) = \left(\frac{1}{s(s+2)} \right)$$

Solution:

- Number of poles = 2, number of zeros = 0.
- Open-Loop poles at {0, -2}. Mark an 'X' at these points.
- Real Axis Loci: (0, -2)
- Number of asymptotes = 2
- Asymptote Angles = $\pm 90^\circ$ (note that the asymptotes split the s-plane into 2 equal regions)
- Asymptote Intercept: $\left(\frac{\sum \text{poles} - \sum \text{zeros}}{n-m} \right) = \left(\frac{-2}{2} \right) = -1$
- Breakaway Point: $\frac{d}{ds}(s(s+2)) = 2s+2 = 0$. $s=-1$.
- $j\omega$ crossings: $s=j0$.

The following is then the root locus plot:



Problem 1b: Describe how the previous system behaves as k increases from 0 to infinity:

Solution: From the root-locus, the closed loop poles start at the open-loop poles and move along the root locus.

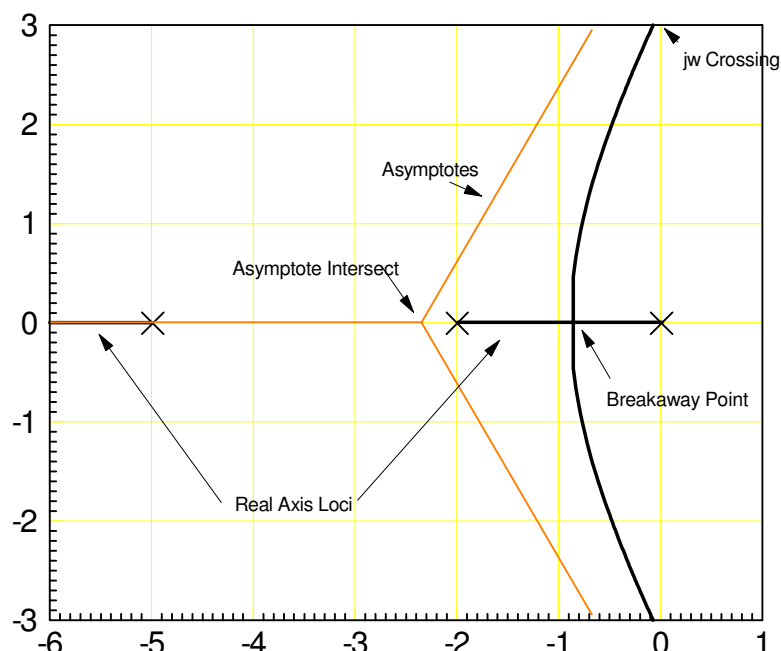
- For k small, the system behaves as a slow first-order system.
- As k increases, the system speeds up until it has a settling time of 4 seconds. (the pole at $s=0$ moves left to $s=-1$).
- When k becomes bigger than 1, the system has a settling time of 4 but starts to have some overshoot in the step response. (The real part of the dominant pole is at -1 but the complex part increases.)
- For large k , the system becomes more and more oscillatory but still has a 4 second settling time.

Problem #2a: Sketch the root locus of

$$G(s) = \left(\frac{1}{s(s+2)(s+5)} \right)$$

Solution: This will be similar to the previous root locus only with a third pole at -5. This pole repels the root locus, pushing it into the right-half plane.

- Number of poles = 3, number of zeros = 0.
- Open-Loop poles at {0, -2, -5}. Mark an 'X' at these points.
- Real Axis Loci: (0, -2), (-5, $-\infty$)
- Number of asymptotes = 3
- Asymptote Angles = $\pm 60^\circ, 180^\circ$ (note that the asymptotes split the s-plane into 3 equal regions)
- Asymptote Intersect: $\left(\frac{\sum \text{poles} - \sum \text{zeros}}{n-m} \right) = \left(\frac{(0)+(-2)+(-5)}{3-0} \right) = -2.3333$
- Breakaway Point: $\frac{d}{ds}(s(s+2)(s+5)) = \frac{d}{ds}(s^3 + 7s^2 + 10s) = 3s^2 + 14s + 10 = 0$ ($s = -0.88, = 3.79$) The point ($s = -0.88$) is on the root locus and is the breakaway point.
- jw crossings: $s = j3.16$. ($G(j3.16) = 0.0143 \angle 180^\circ$)
- The following is then the root locus plot



Note: The third pole at -5 has

- Pushed the root-locus to the right (causing it to go into the right-half plane)
- Pushed the breakaway point away from -1 (the midpoint between the former breakaway point)

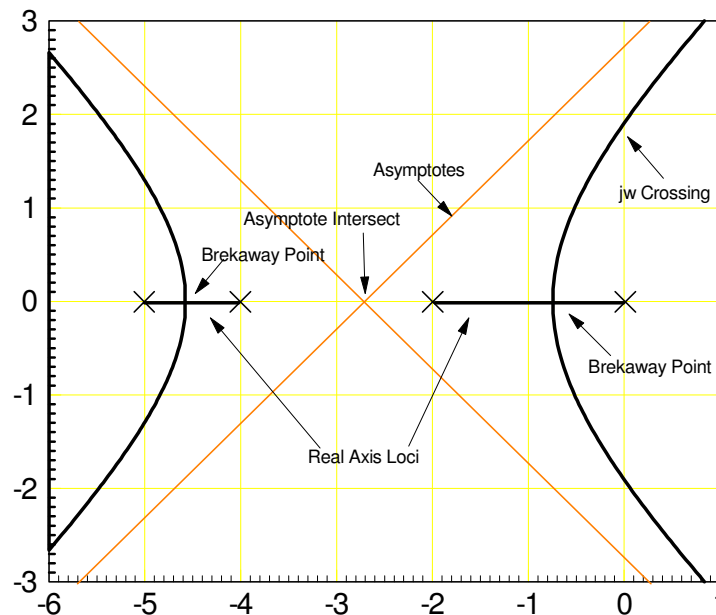
Example 3: Sketch the root locus of

$$G(s) = \left(\frac{1}{s(s+2)(s+4)(s+5)} \right)$$

Solution:

- Number of poles = 4, number of zeros = 0.
- Open-Loop poles at {0, -2, -4, -5}. Mark an 'X' at these points.
- Open-Loop zeros: none
- Real Axis Loci: (0, -2), (-4, -5)
- Number of asymptotes = 4
- Asymptote Angles = $\pm 45^\circ, \pm 135^\circ$
- Asymptote Intersect: $\frac{(0)+(-2)+(-4)+(-5)}{4} = -2.75$
- Breakaway Point: $\frac{d}{ds} \left(\frac{s(s+2)(s+5)(s+10)}{s^2+2s+2} \right) = 0 \quad s = \{-0.7438, -4.5771\}$
- $j\omega$ crossings: $s = j1.9069 \quad G(j1.9069) = 0.0080 \angle 180^\circ$

The following is then the root locus plot

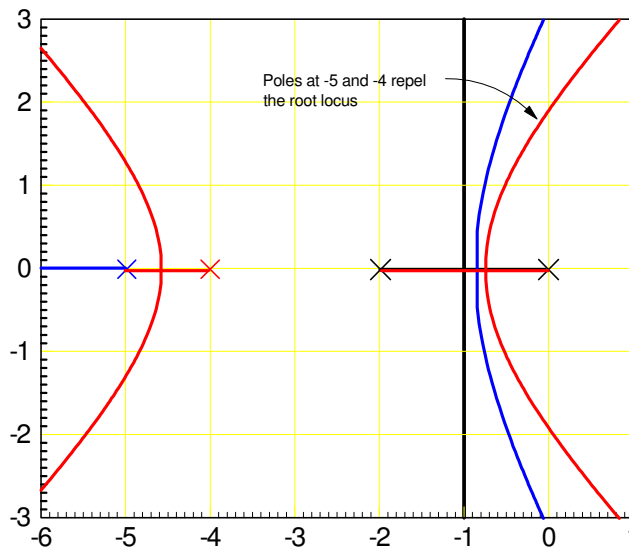


Note: The two pole at -4, and -5 have

- Pushed the root locus to the right even harder (now going right at an angle of 45 degrees)
- Pushed the breakaway point right of -1 (the former breakaway point).

Summary:

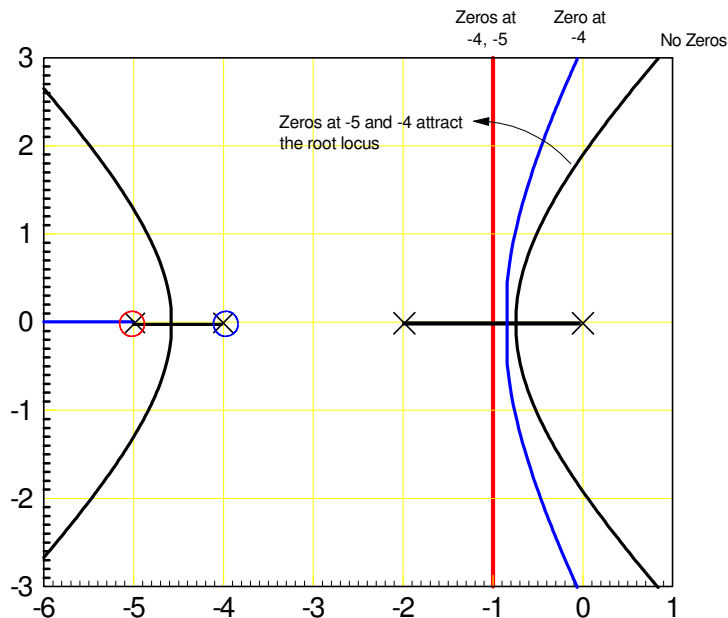
Note that poles repel the root locus. As you add poles at -4 and -5, the root locus is pushed further and further to the right.



Zeros behave the opposite way of poles: they attract the root locus. For example, if you compare

$$G(s) = \left(\frac{1}{s(s+2)(s+4)(s+5)} \right) \text{ vs } \left(\frac{(s+4)}{s(s+2)(s+4)(s+5)} \right) \text{ vs } \left(\frac{(s+4)(s+5)}{s(s+2)(s+4)(s+5)} \right)$$

the zeros at -4 and -5 will pull the roots towards them as shown below.



This will be a useful tool later on:

- If you want to push the root-locus to the right, add a pole to the left or slide fast pole right.
- If you want to pull a root-locus to the left, add zeros to the left to attract the root-locus.