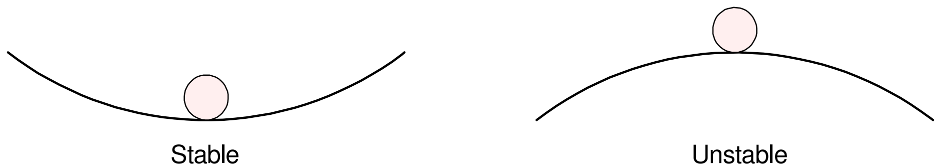
Routh Criteria & Stability

Stability is defined as a system returning to its original state when perturbed. For example, a ball at the bottom of a ravine is stable: if you perturb it, it will return to its original spot. In contrast, a ball on top of a hill is unstable: if you pertub it, it will roll away



Stability: If perturbed, it will return to its original position

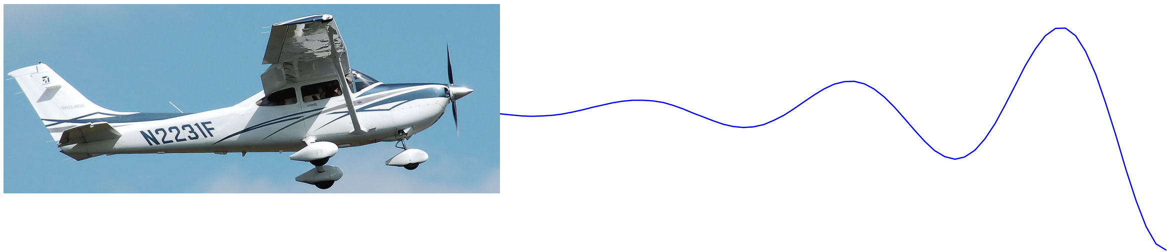
Pretty much, for a system to be useful, it has to be stable: unstable systems tend to break (or blow up).

Feedback is a very useful tool. With feedback, you can take systems which are unstable and make them stable (and useful). For example

- Standing is unstable: if you pass out (i.e. you remove the feedback loop), you fall down.
- Bicycles are unsable: it takes days and days of practice (and a few skinned knees) to learn how to ride a bike.
- Rockets are unsable: they are essentially a pendulum balancing on a ball of flame. Turn off the feedback and a rocket spins out of control. With feedback, you can launch a rocket, put people into space, and bring them back in one piece.
- Economies are unstable: during boom times companies sell more, hire more people, who in turn buy more. In bad times, people quit buying products, comanies start losing money and lay off more people, who in turn buy even less. This pattern was noted as far back as the Roman Empire. The Federal Reserve's job is to adjust interest rates and the money supply to stabilize the economy and keep the growth rate at a constant, sustailable level.

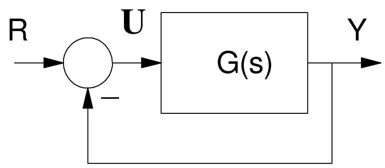
One problem with feedback is you can take a perfectly stable system and make it unstable just by adding too much feedback. For example, when you first learned how to drive, you probably jerked too hard on the steering wheel sending the car towards the curb. You then over-corrected, jerked the steering wheel, sending the car towards the other curb. Eventually, you learn how apply the correct amount of feedback to keep the car on the road and in your lane.

This also shows up with aircraft with a thing called "pilot induced oscillations." If you take your hands off a Cessna, the airplane will fly itself: a Cessna is designed to be open-loop stable. Sometimes, a pilot will start over-correcting, sending the airplane towards the ground and then veering off the sky. To fix this problem you take you hands off the controls (i.e. remove the feedback loop) and let the airplane fly itself.



Pilot-Induced Oscillations: Wild pitching in an aircraft due to the pilot over-correcting with the controls

To illistrate this, consider the following feedabck loop



where G(s) is a nice stable system (think 3-stage RC filter or a heat equation).:

$$Y = \left(\frac{100}{(s+1)(s+2)(s+3)}\right)U$$

With feedback, the transfer function becomes

$$Y = \left(\frac{G}{1+G}\right)R$$

$$Y = \left(\frac{100}{(s+1)(s+2)(s+3)+100}\right)R$$

or factoring

$$Y = \left(\frac{100}{(s-0.3567+j3.9575)(s-0.3567-j3.9575)(s+6.7134)}\right)R$$

Note that the closed-loop system is unstable in spite of the open-loop system being a well behaved stable system.

So, this brings up two questions:

- How do you determine whether a system is stable?
- How do you determine the range of feedback gains that result in a stable system?

The forrmer question is somewhat easy: put the transfer function into Matlab and find the roots (i.e. the eigenvalues).

- If all of the eigenvalues are negative definite, the system is stable.

- If one or more of the eigenvalues are positive real, the system is unstable.

It only takes one unstable pole to make the whole system unstable.

The latter question is harder but useful:

- Feedback control systems often have knobs you can adjust. It would be nice if you could dummy-proof such a system and make sure the operator doesn't make the system go unsable.
- Sometimes, a system just isn't stabizable. It would be nice to know this before spending hours trying to stabilize it.

Routh Tables and the Routh Criteria is one tool which allows you to do this:

- Determine if a given system is stable, and
- Determine the range of gains which result in a stable system.

Routh Tables & Routh Criteria

Determine if a given polynomial

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0 = 0$$

is negative definite.

Step 1: Set up a Routh Table.

1) The first two rows take the even and coefficients of the polynomial and odd terms respectively:

a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...

2) Using the two previous rows, create the next row as

Previous two rows:	a	b	c	d	...
	e	f	g	h	...
New Row	$\frac{-\begin{vmatrix} a & b \\ e & f \end{vmatrix}}{e}$	$\frac{-\begin{vmatrix} a & c \\ e & g \end{vmatrix}}{e}$	$\frac{-\begin{vmatrix} a & d \\ e & h \end{vmatrix}}{e}$	etc.	

3) Repeat step 2 for the next row. Repeat until all entries are zero.

Step 2: Count the number of sign flips in column #1. The number of sign flips is equal to the number of positive (unstable) roots to this polynomial.

- **Routh Criteria:** There can be no sign flips in column #1 in order for a system to be stable.

Example: Determine if

$$(s+1)(s+2)(s+3)(s+4) = s^4 + 10s^3 + 35s^2 + 50s + 24$$

has any unstable roots.

Solution: Set up the Routh table. The first two rows are

1	35	24	0	0
10	50	0	0	0

The next rows are

1	35	24	0
10	50	0	0
$\frac{-\begin{vmatrix} 1 & 35 \\ 10 & 50 \end{vmatrix}}{10} = 30$	$\frac{-\begin{vmatrix} 1 & 24 \\ 10 & 0 \end{vmatrix}}{10} = 24$	$\frac{-\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0$	0
$\frac{-\begin{vmatrix} 10 & 50 \\ 30 & 24 \end{vmatrix}}{30} = 42$	$\frac{-\begin{vmatrix} 10 & 0 \\ 30 & 0 \end{vmatrix}}{30} = 0$	$\frac{-\begin{vmatrix} 10 & 0 \\ 30 & 0 \end{vmatrix}}{30} = 0$	
$\frac{-\begin{vmatrix} 30 & 24 \\ 42 & 0 \end{vmatrix}}{42} = 24$	$\frac{-\begin{vmatrix} 30 & 0 \\ 42 & 0 \end{vmatrix}}{42} = 0$		
$\frac{-\begin{vmatrix} 42 & 0 \\ 24 & 0 \end{vmatrix}}{24} = 0$	$\frac{-\begin{vmatrix} 42 & 0 \\ 24 & 0 \end{vmatrix}}{24} = 0$		

There are no sign flips. Hence, this polynomial is stable. (The system is stable).

Example 2: Determine the range of K for which the following system is stable:

$$Y = \left(\frac{K}{(s+1)(s+2)(s+3)+K}\right)R$$

Solution:

i) Factor the denominator polynomial:

$$(s+1)(s+2)(s+3)+K = s^3+6s^2+11s+6+K$$

ii) Insert the coefficients into the first two rows of a Routh Table:

1	11	0	
6	6+K	0	

iii) Fill in the rest of the table:

1	11	0	
6	6+K	0	
$\frac{60-K}{6}$	0	0	

One property of Routh Tables is you can scale any row by a positive constant and not change the results. Multiplying by 6 to keep the numbers pretty:

1	11	0	
6	6+K	0	
60-K	0	0	
6+K	0	0	
0	0	0	

iv) Select K so that there are *no* sign flips in column #1:

1	11	0	
6	6+K	0	
60-K	0	0	K < 60
6+K	0	0	K > -6
0	0	0	

The valid range of K will be those values which satisfy every condition in the right column:

(-6 < K < 60)

Note: at K = -6, the roots to $s^3 + 6s^2 + 11s + 6 + K$ are {0, -3+j1.4142, -3-j1.4142}. (One pole is on the jw axis)

At K=+60, the roots are {-6, +j3.3166, -j3.3166}. (Two poles are on the jw axis)

Example 3: Suppose you add a PID compensator to the previous system

$$K(s) = \left(P + \frac{I}{s} + Ds\right)$$

Determine the range of PID so that the closed-loop system is stable:

The open-loop system is:

$$GK = \left(\frac{1}{(s+1)(s+2)(s+3)}\right)\left(P + \frac{I}{s} + Ds\right)$$

$$GK = \left(\frac{Ds^2+Ps+I}{s(s+1)(s+2)(s+3)}\right)$$

The closed-loop system is:

$$\left(\frac{GK}{1+GK}\right) = \left(\frac{Ds^2+Ps+I}{s(s+1)(s+2)(s+3)+\left(Ds^2+Ps+I\right)}\right)$$

To be stable, the denominator polynomial must be negative definite, or the Routh Table is applied to the following polynomial:

$$s(s+1)(s+2)(s+3) + (Ds^2 + Ps + I) = 0$$

Multiplying this out

$$(s^4+6s^3+11s^2+6s)+(Ds^2+Ps+I)=0$$

$$s^4+(6)s^3+(11+D)s^2+(6+P)s+(I)=0$$

Set up the Routh Table:

1	11+D	I
6	6+P	0
$10+D-\frac{P}{6}$	I	0
$\frac{60+6D+P+10P+DP+\frac{P^2}{6}-6I}{10+D+\frac{P}{6}}$	0	0
I	0	0

$$\frac{-\begin{vmatrix}1 & 11+D \\ 6 & 6+P\end{vmatrix}}{6}=\frac{(66+6D)-(6+P)}{6}=10+D-\frac{P}{6}$$

$$\frac{-\begin{vmatrix}6 & 6+P \\ 10+D+\frac{P}{6} & I\end{vmatrix}}{10+D+\frac{P}{6}}=\frac{\left(10+D+\frac{P}{6}\right)(6+P)-6I}{10+D+\frac{P}{6}}=\frac{60+6D+P+10P+DP+\frac{P^2}{6}-6I}{10+D+\frac{P}{6}}$$

For the closed-loop system to be stable:

$$10+D-\frac{P}{6}>0$$

$$60+6D+P+10P+DP+\frac{P^2}{6}-6I>0$$

$$I>0$$

This is a result which is difficult to obtain using other methods.