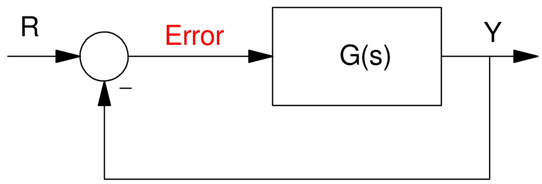
Error Constants and Steady-State Error

Now that we can model different dynamic systems, let's look at how these systems behave when using output feedback.



Feedback Configuration: Ideally, Y = R (meaning the error is zero)

Error constants attempt to assign a single number to a system to assess how "good" it is

- Ideally, the error should be zero (meaning Y = R, the output is following the reference signal)

In addition, it should track different types of inputs

- R = 1 (unit step) models tracking a constant set point
- R = t (unit ramp) models tracking a set point with constant velocity
- R = t^2 (unit parabola) models tracking a set point with constant acceleration

Error constants and steady-state error are way to describe the steady-state behavior of a dynamic system with these set points.

Definitions:

Type N System: The plant, G(s), has N poles at s=0.

Error Constants: Kp, Kv, Ka. The DC gain of a plant:

Kp: The DC gain of a type-0 system

- $K_p = \lim_{s \to 0} (G(s))$
- $\lim_{s \to 0} (G(s)) = K_p$

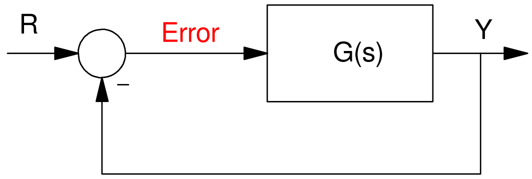
Kv: The DC gain of a type-1 system:

- $K_v = \lim_{s \to 0} (s \cdot G(s))$
- $\lim_{s \to 0} (G(s)) = \frac{K_v}{s}$

Ka: The DC gain of a type-2 system:

- $K_a = \lim_{s \to 0} (s^2 \cdot G(s))$
- $\lim_{s \to 0} (G(s)) = \frac{K_z}{s^2}$

Steady-State Error: The steady-state error is the difference between the desired output and actual output (R-Y) for a unity feedback system:



Steady-State Error: The error as t goes to infinity

Three standard inputs are used to test how well a system behaves:

Unit Step:

- $R(s) = \frac{1}{s}$
- Historically modeled trying to point an antiaircraft gun at a German bomber in WWII.
- Models tracking a constant setpoint, such as holding the temperature of a room at 72F, regulating the flow through a pipe to 0.1m3/s, holding the speed of a car at 55mph, etc.

Unit Ramp:

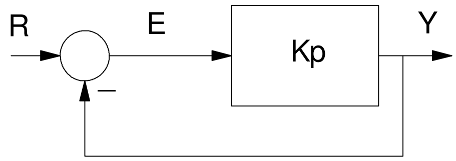
- $R(s) = \frac{1}{s^2}$
- Historically modeled trying to track a German bomber moving at a constant speed across the sky.
- Models tracking a setpoint which is rising or falling at a constant rate.

Unit Parabola

- $R(s) = \frac{1}{s^3}$
- Historically, models German bombers as they fly over you (the angle of the antiaircraft gun whips around at the zenith).
- I'm not sure what this models.

Type-0 Systems:

Assume that at DC G(s)=Kp.



The error wil then be

$$Y = \left(\frac{K_p}{K_p+1}\right)R$$

$$E = R - Y = \left(\frac{1}{K_p+1}\right)R$$

i) if R(s) is a unit step:

$$E = \left(\frac{1}{K_p+1}\right)\left(\frac{1}{s}\right)$$

$$e = \left(\frac{1}{K_p+1}\right) \quad \text{(t>0)}$$

A type-0 system will have a constant error of $\left(\frac{1}{K_p+1}\right)$ when tracking a unit step input

ii) if R(s) is a unit ramp:

$$E(s) = \left(\frac{1}{K_p+1}\right)\left(\frac{1}{s^2}\right)$$

$$e(t) = \left(\frac{1}{K_p+1}\right)t$$

As $t \to \infty$, the error goes to ∞.

A type-0 system cannot track a unit ramp input.

Example: Find the steady-state error for

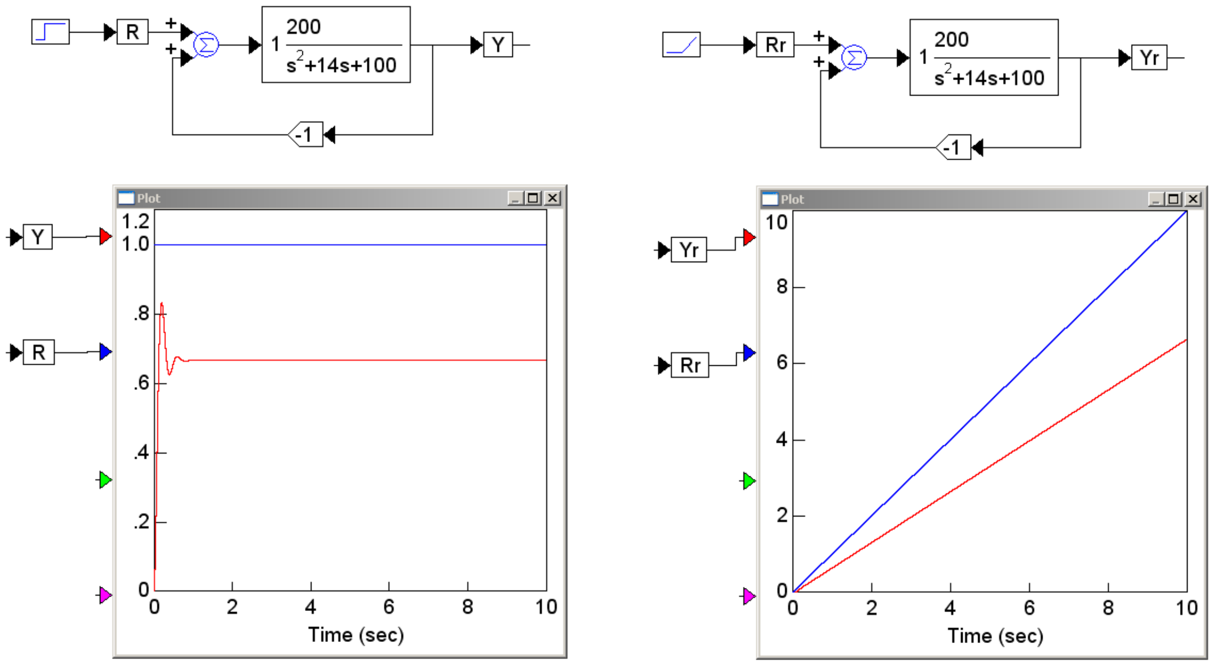
$$G(s) = \left(\frac{200}{s^2+14s+100}\right)$$

Solution: This is a type-0 system (there are no poles at s = 0). The error for a step input is

$$K_p = G(s)_{s \to 0} = 2$$

$$E_{step} = \frac{1}{K_p+1} = \frac{1}{3}$$

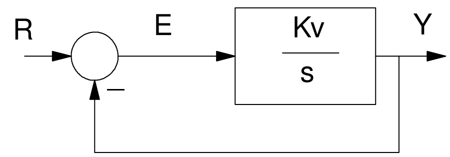
$$E_{ramp} = \infty$$



Type-0 Systems Have Constant Error for a Step Input (left) and infinite error for a ramp input (right)

Type-1 Systems:

Assume at DC $G(s) = \frac{K_v}{s}$:



Then

$$Y = \left(\frac{\frac{K_v}{s}}{\frac{K_v}{s}+1}\right)R = \left(\frac{K_v}{K_v+s}\right)R$$

$$E = R - Y = \left(\frac{s}{K_v+s}\right)R$$

i) R = a unit step:

$$E = \left(\frac{s}{K_v+s}\right)\left(\frac{1}{s}\right) = \left(\frac{0}{s}\right) + \left(\frac{1}{K_v+s}\right)$$

$$e(t) = 0 + transient \quad \text{(t>0)}$$

$$e(\infty) = 0$$

A type-1 system can track a constant set point with no error.

ii) R = a unit ramp:

$$E = \left(\frac{s}{K_v+s}\right)\left(\frac{1}{s^2}\right) = \left(\frac{1}{K_v+s}\right)\left(\frac{1}{s}\right) = \left(\frac{\frac{1}{K_v}}{s}\right) + \left(\frac{c}{s+K_v}\right)$$

$$e(t) = \left(\frac{1}{K_v}\right) + transients$$

$$e(\infty) = \left(\frac{1}{K_v}\right)$$

A type-1 system can track a unit ramp with a constant error of $\left(\frac{1}{K_v}\right)$

iii) R = a unit parabola:

$$E = \left(\frac{s}{K_v+s}\right)\left(\frac{1}{s^3}\right) = \left(\frac{1}{K_v+s}\right)\left(\frac{1}{s^2}\right) = \left(\frac{\frac{1}{K_v}}{s^2}\right) + \left(\frac{c}{s}\right) + \left(\frac{d}{s+K_v}\right)$$

$$e = \left(\frac{1}{K_v}\right)t + \ldots$$

$$e(\infty) = \infty$$

A type 1 system cannot track a unit parabola.

Example: Find the steady-state error for

$$G(s) = \left(\frac{200}{s\left(s^2+14s+100\right)}\right)$$

Solution: This is a type-1 system (there is one pole at s = 0). The error constant is

$$G(s)_{s\to 0} = \frac{2}{s}$$

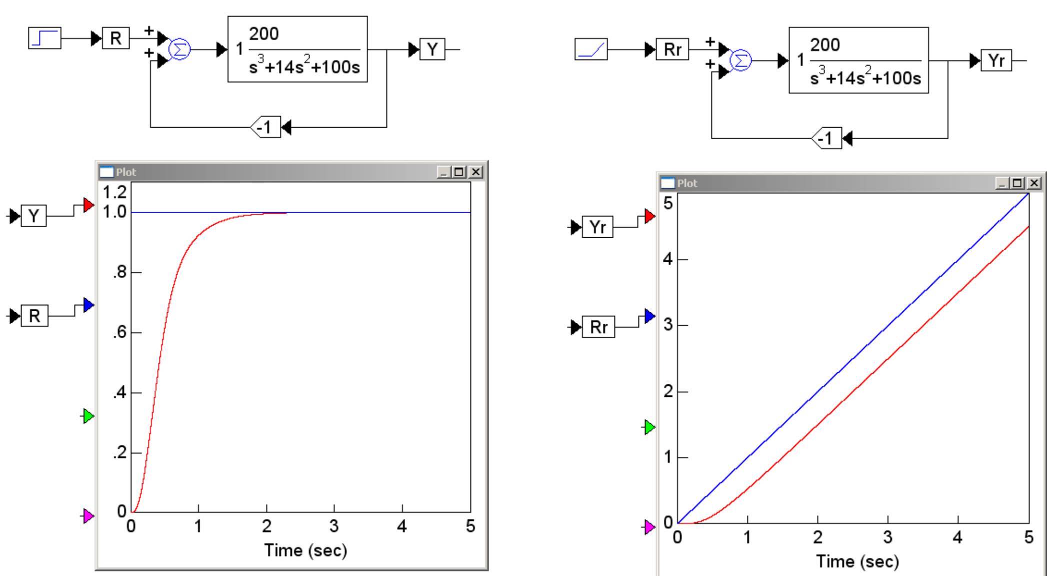
Kv = 2. The error for a step input is zero (type-1 system track constant set points with no error).

$$E_{step} = 0$$

The error for a unit ramp input is

$$E_{ramp} = \frac{1}{K_v} = \frac{1}{2}$$

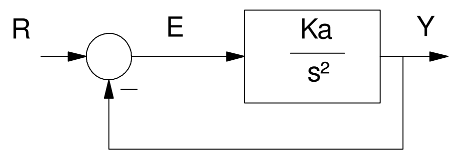
In VisSim:



Type-1 systems have no error for a step input (left) and constant error for a ramp input (right)

Type-2 systems:

Assume G(s) is a type-2 system:



In this case

$$Y = \left(\frac{\frac{K_a}{s^2}}{\frac{K_a}{s^2}+1} \right) R = \left(\frac{K_a}{K_a+s^2} \right) R$$

$$E = R - Y = \left(\frac{s^2}{K_a+s^2} \right) R$$

i) If R is a unit step:

$$E = \left(\frac{s^2}{K_a+s^2} \right) \frac{1}{s} = 0 + \left(\frac{c}{s^2+K_a} \right)$$

$$e(t) = 0 + (transients)$$

$$e(\infty) = 0$$

A type-2 system can track a constant set-point with no error.

ii) if R is a unit ramp:

$$E = \left(\frac{s^2}{K_a+s^2} \right) \frac{1}{s^2} = 0 + \left(\frac{1}{s^2+K_a} \right)$$

$$e(t) = 0 + (transients)$$

$$e(\infty) = 0$$

A type-2 system can track a ramp input with no error

iii) Unit parabola:

$$E = \left(\frac{s^2}{K_a+s^2} \right) \frac{1}{s^3} = \left(\frac{\frac{1}{K_z}}{s} \right) + \left(\frac{c}{s^2+K_a} \right)$$

$$e(t) = \left(\frac{1}{K_z} \right) + transients$$

$$e(\infty) = \left(\frac{1}{K_a}\right)$$

A type-2 system can track a unit parabola with a constant error of $\left(\frac{1}{K_a}\right)$

Example: Determine the steady-state error for a step and ramp input for the following system:

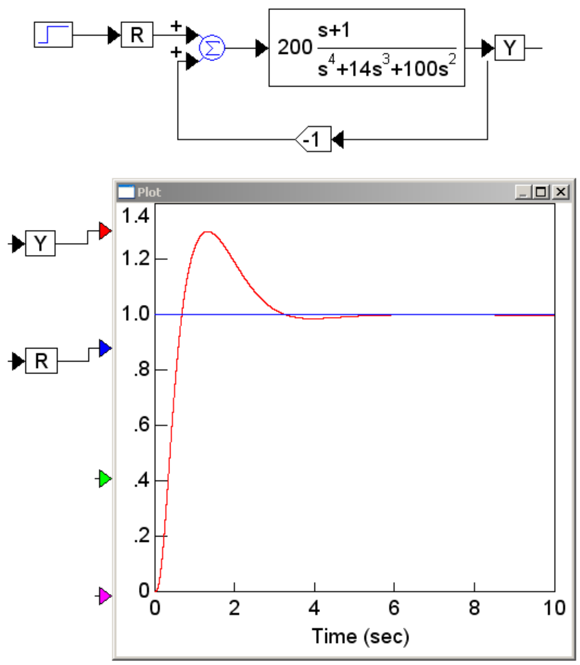
$$G(s) = \left(\frac{200(s+1)}{s^2\left(s^2+14s+100\right)}\right)$$

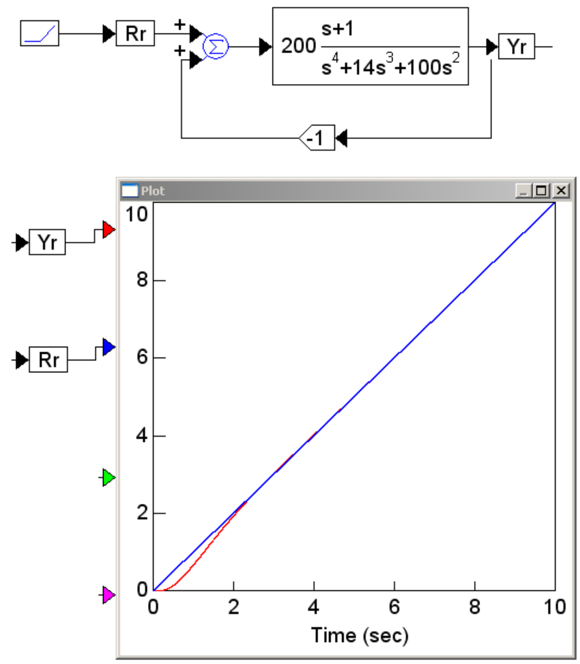
Solution: This is a type-2 system (there are two poles at s = 0) with Ka = 2.

$$G(s)_{s \to 0} = \frac{2}{s^2} = \frac{K_a}{s^2}$$

Being a type-2 system, the steady-state error for a step and ramp input will be zero.

In VisSim:





Type 2 systems have no error for a step input or a ramp input (assuming it's stable).

Summary:

Computation of Error Constants			
	Kp	Kv	Ka
Type 0	$\lim_{s \to 0} (G(s))$	0	0
Type 1	∞	$\lim_{s \to 0} (s \cdot G(s))$	0
Type 2	∞	∞	$\lim_{s \to 0} (s^2 \cdot G(s))$

Steady-State Error			
	Unit Step	Unit Ramp	Unit Parabola
Type 0	$\left(\frac{1}{K_p+1}\right)$	∞	∞
Type 1	0	$\left(\frac{1}{K_v}\right)$	∞
Type 2	0	0	$\left(\frac{1}{K_a}\right)$