Rotational Systems & Gears

Rotational Systems

Rotational systems behave exactly like translational systems, except that

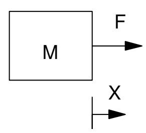
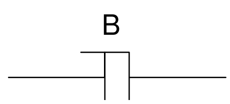
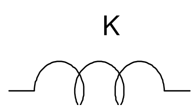
- The state (angle) is denoted with θ rather than x (position)
- Inertia is called J (rather than M)
- Friction is called D (rather than B)

Otherwise, the symbols, circuit equivalent, and analysis for rotational systems is exactly like translational systems.

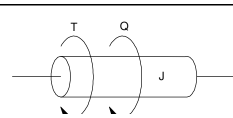
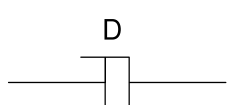
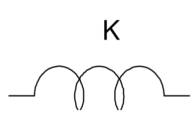
Electrical Circuits

Z = R (resistor) Z = Ls (inductor) Z = 1/Cs (capacitor)	$I = \frac{1}{Z}V$

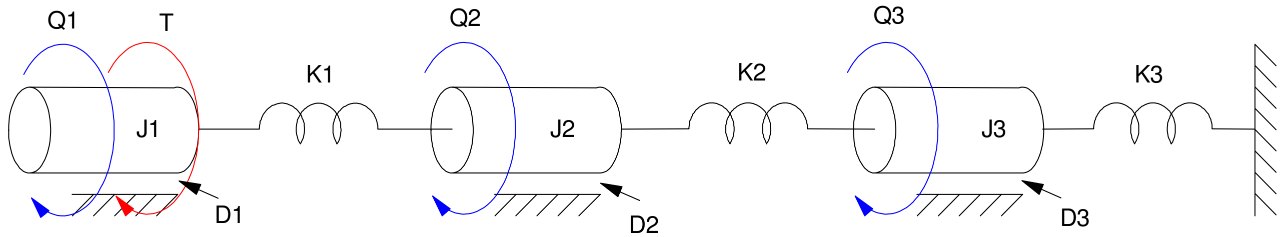
Translational Systems

Inertia		$F = Ms^2X$
Friction		$F = BsX$
Spring		$F = KX$

Rotational Systems

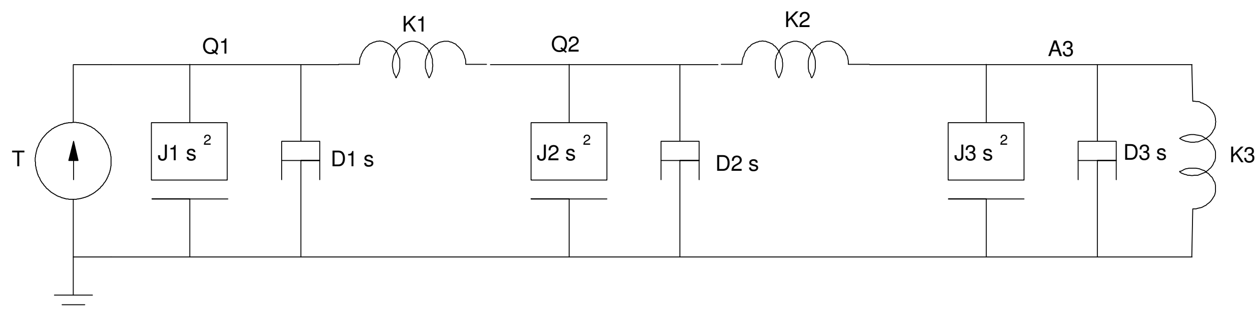
Inertia		$T = Js^2\theta$
Friction		$T = Ds\theta$
Spring		$T = K\theta$

Example: Draw the circuit equivalent and write the equations of motion for the following rotational system.



Inertia - Spring Rotational System

Solution: Note that each term is an admittance. Draw the circuit equivalent. There are three masses, which result in three voltages nodes (each angle corresponds to a voltage node). Each element corresponds to an impedance.



Circuit Equivalent

The voltage node equations are then:

$$(J_1 s^2 + D_1 s + K_1)\theta_1 - (K_1)\theta_2 = T$$

$$(J_2 s^2 + D_2 s + K_1 + K_2)\theta_2 - (K_1)\theta_1 - (K_2)\theta_2 = 0$$

$$(J_3 s^2 + D_3 s + K_2 + K_3)\theta_3 - (K_2)\theta_2 = 0$$

Placing this in state-space form is identical to what we did with mass-spring systems. Solve for the highest derivative:

$$s^2\theta_1 = -\left(\frac{D_1}{J_1}s + \frac{K_1}{J_1}\right)\theta_1 + \left(\frac{K_1}{J_1}\right)\theta_2 + \left(\frac{1}{J_1}\right)T$$

$$s^2\theta_2 = -\left(\frac{D_2 s}{J_2} + \frac{K_1+K_2}{J_2}\right)\theta_2 + \left(\frac{K_1}{J_2}\right)\theta_1 + \left(\frac{K_2}{J_2}\right)\theta_2$$

$$s^2\theta_3 = -\left(\frac{D_3 s}{J_3} + \frac{K_2+K_3}{J_3}\right)\theta_3 + \left(\frac{K_2}{J_3}\right)\theta_2$$

Defining the states to be the angles and the derivatives, in matrix form this is

$$\begin{bmatrix} s\theta_1 \\ s\theta_2 \\ s\theta_3 \\ \cdots \\ s^2\theta_1 \\ s^2\theta_2 \\ s^2\theta_3 \end{bmatrix} = \left|\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{-K_1}{J_1} & \frac{-K_1}{J_1} & 0 & \frac{-D_1}{J_1} & 0 & 0 \\ \frac{K_1}{J_1} & \frac{-K_1-K_2}{J_1} & \frac{K_2}{J_1} & 0 & \frac{-D_2}{J_2} & 0 \\ 0 & \frac{K_2}{J_1} & \frac{-K_2-K_3}{J_1} & 0 & 0 & \frac{-D_3}{J_3} \end{array}\right| \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \cdots \\ s\theta_1 \\ s\theta_2 \\ s\theta_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} T$$

Gears

Gears are like transformers:

- They convert one speed to another as the gear ratio,
- They convert one impedance to another as the gear ratio squared.

The governing equation for a gear is that the distance traveled at the interface must be the same for both gears:

$$\text{distance} = r\theta$$

$$d_1 = r_1\theta_1$$

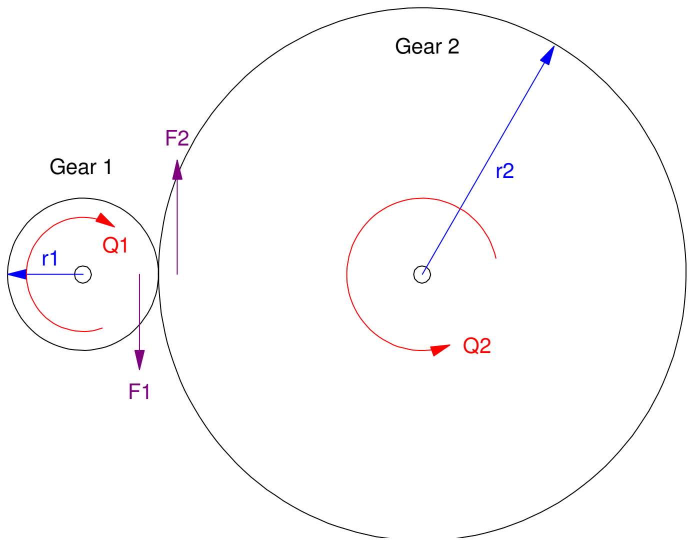
$$d_2 = r_2\theta_2$$

$$d_1 = d_2$$

$$r_1\theta_1 = r_2\theta_2$$

or

$$\theta_1 = \left(\frac{r_2}{r_1}\right)\theta_2$$



Two gears, where gear 1 turns gear 2
The gear which is 4x smaller will turn 4x faster

The torques are also related by the turn ration. Torque is:

$$T_1 = F_1 r_1$$

$$T_2 = F_2 r_2$$

At the tip, each force has an equal and opposite force:

$$F_1 = F_2$$

resulting in

$$\frac{T_1}{r_1} = \frac{T_2}{r_2}$$

$$T_1 = \left(\frac{r_1}{r_2}\right)T_2$$

Admittance is the ratio

$$T_1 = Ms^2\theta_1$$

Relative to the other side of the gear:

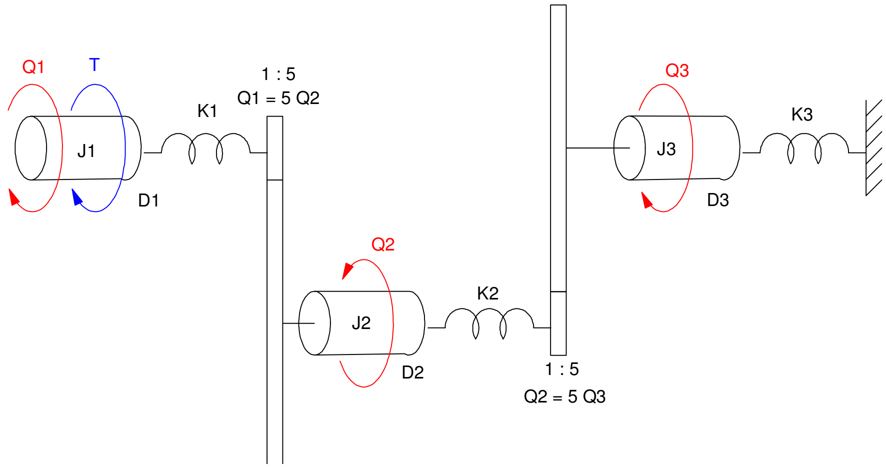
$$\left(\frac{r_1}{r_2}\right)T_2 = (Ms^2)\left(\frac{r_2}{r_1}\right)\theta_2$$

or

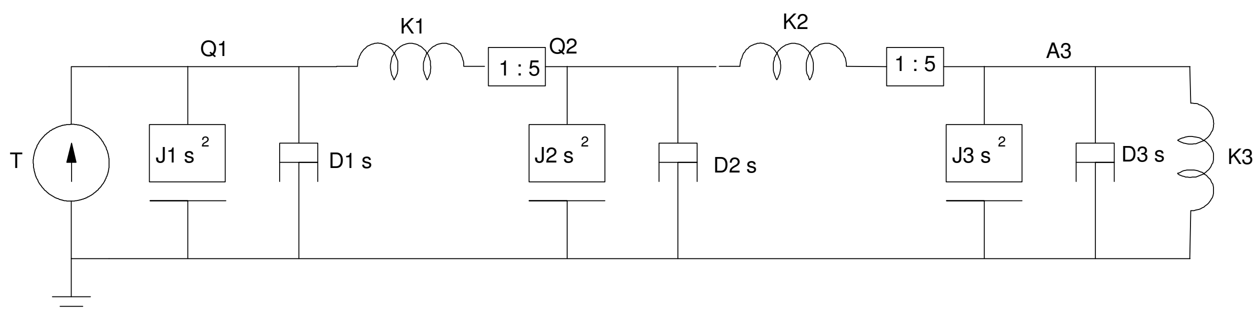
$$T_2 = \left(\frac{r_2}{r_1}\right)^2(Ms^2)\theta_2$$

Impedances go through a gear as the turn ratio squared

Example:



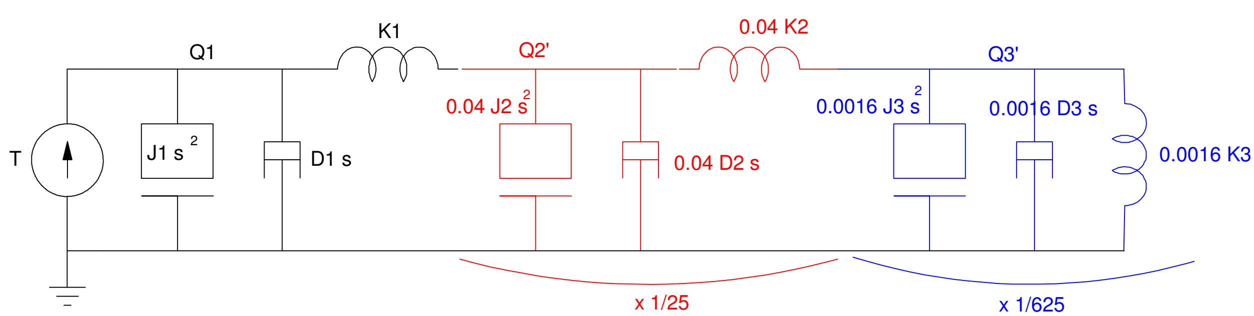
i) Draw the circuit equivalent with the gears



Circuit Equivalent for Rotational Gear System

ii) Take out the gears by scaling the impedance by the turn ratio squared

$$Y_{new} \Rightarrow \left(\frac{\text{where going to}}{\text{where coming from}}\right)^2 Y_{old}$$



Circuit equivalent with the gears (transformers) removed

iii) Write the equations of motion (i.e. write the voltage node equations, noting that all terms are impedance). (note: ϕ is used rather than θ as a reminder that the node voltages are scaled by the gear ratio.)

$$(J_1 s^2 + D_1 s + K_1)\phi_1 - (K_1)\phi_2 = T$$

$$\left(\frac{J_2}{5^2}s^2 + \frac{D_2}{5^2}s + K_1 + \frac{K_2}{5^2}\right)\phi_2 - (K_1)\phi_1 - \left(\frac{K_2}{5^2}\right)\phi_3 = 0$$

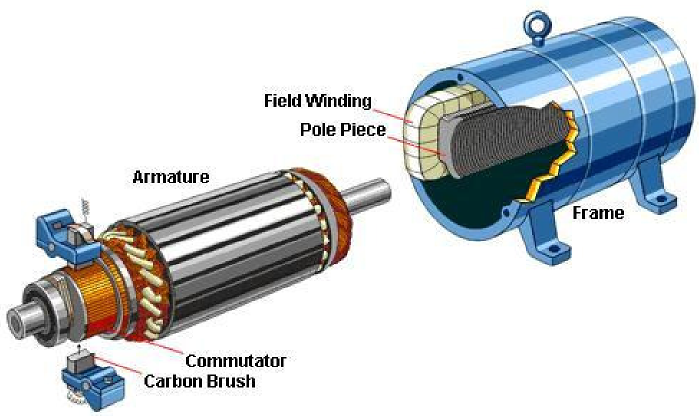
$$\left(\frac{J_3}{25^2}s^2 + \frac{D_3}{25^2}s + K_1 + \frac{K_2}{5^2}\right)\phi_3 - \left(\frac{K_2}{5^2}\right)\phi_2 = 0$$

From this point onward, it's the same as before.

DC Servo Motors

A DC Servo Motor is a type of motor which is often used with rotational systems. It has the advantage that

- Torque is proportional to current (making it easy to apply torques to a rotational system)
- Spins when a constant voltage is applied (operating as a motor), or
- Provides a constant voltage when spun (operating as a DC generator - a.k.a. a dynamo)

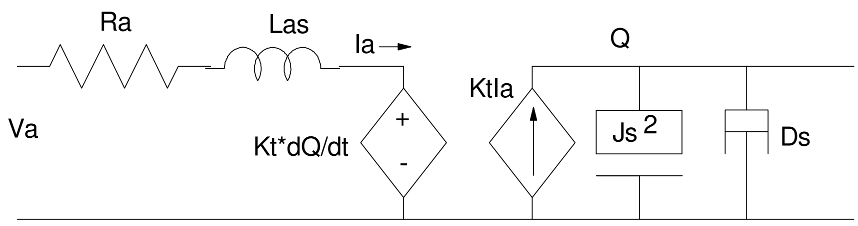


A DC servo motor is a 2-lead motor which spins when a constant voltage is applied (motor) or produces a constant voltage when spun (generator) (image from http://www.electrical-knowhow.com/2012/05/classification-of-electric-motors.html)

Internally, a DC motor has

- A permanent magnet (field winding) on the outside,
- An electromagnet on the inside (armature), and
- A commutator which switches the polarity of the current (voltage) so that the torque remains in the same direction as the motor spins.

A DC servo motor is essentially an electromagnet, with the following circuit equivalent:



Circuit equivalent for a DC servo motor

Kt is a constant for the motor. It describes how the motor works as a generator, creating a back voltage proportional to speed

$$E_a = K_t \frac{d\theta}{dt} = K_t \omega$$

as well as a motor where the torque is proportional to the armature current

$$T = K_t I_a$$

It isn't obvious, but Kt is the same for both equations. To see this, note that power out = power in

$$P_{in} = P_{out}$$

$$E_a I_a = T\omega$$

Substituting for Ea and T:

$$(K_{t1}\omega)I_a = (K_{t_2} I_a)\omega$$

$$K_{t1} = K_{t_2}$$

Here you can see that the two constants are the same.

The equations for this are then

or

$$V_a = K_t(sQ) + (R_a + L_a s)I_a$$

$$K_t I_a = (Js^2 + Ds)Q$$

Solving gives

$$Q = \left(\frac{1}{s}\right)\left(\frac{K_t}{(Js+D)(L_a s+R_a)+K_t^2}\right)V_a$$

if the output is angle (Q) or

$$\omega = \frac{dQ}{dt} = \left(\frac{K_t}{(Js+D)(L_a s+R_a)+K_t^2}\right)V_a$$

if the output is speed.

Example: Determine the transfer function for a DC servo motor with the following specifications:

(http://www.servosystems.com/electrocraft_dcbrush_rdm103.htm)

- $K_t = 0.03 Nm/A$
- Terminal resistance = 1.6 (Ra)
- Armature inductance = 4.1mH (La)
- Rotor Inertia = 0.008 oz-in/sec/sec *ugh - english units*
- Damping Contant = 0.25 oz-in/krpm *double ugh*
- Torque Constant = 13.7 oz-in/amp
- Peak current = 34 amps
- Max operating speed = 6000 rpm

You need to translate to metric:

Kt: $$13.7\frac{oz-in}{amp}\left(\frac{1m}{39.4in}\right)\left(\frac{1lb}{16oz}\right)\left(\frac{0.454kg}{lb}\right)\left(\frac{9.8N}{kg}\right) = 0.0967\left(\frac{N\cdot m}{A}\right)$$

D: $$0.25\left(\frac{oz\cdot in\cdot \min}{krev}\right)\left(\frac{1Nm}{141.7oz-in}\right)\left(\frac{60\sec}{\min}\right)\left(\frac{krev}{1000rev}\right)\left(\frac{rev}{2\pi rad}\right) = 1.68\cdot 10^{-5}\left(\frac{Nm}{rad/\sec}\right)$$

J: $$0.008\left(\frac{oz-in}{\sec^2}\right)\left(\frac{Nm}{141.7oz-in}\right)\left(\frac{kg}{9.8N}\right) = 5.76\cdot 10^{-6}\left(\frac{kg\cdot m}{s^2}\right)$$

Plugging in numbers:

$$Q = \left(\frac{10.31\cdot 630^2}{s(s^2+393s+630^2)}\right)V = \left(\frac{10.31(630)^2}{s(s+196.5+j598)(s+196.5-j598)}\right)V$$

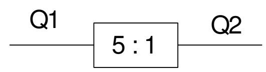
This means....

- If you apply a constant +12V to the motor, the motor spins at a speed of 123.72 rad/sec. (10.31*12)
- It takes about 20ms for the motor to get up to speed (Ts = 4/196)

The motor will overshoot it's final speed (123 rad/sec) by 35% (damping ratio = 0.3119)

Gears and Wheels:

It isn't obvious, but a wheel is a type of gear or transformer. The circuit symbol for a gear



Symbol for a gear: 5Q1 = Q2

means

$$5 \cdot \theta_1 = 1 \cdot \theta_2$$

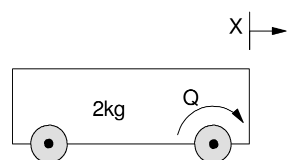
A wheel converts rotational motion to translational motion as

$$x = r\theta$$

where

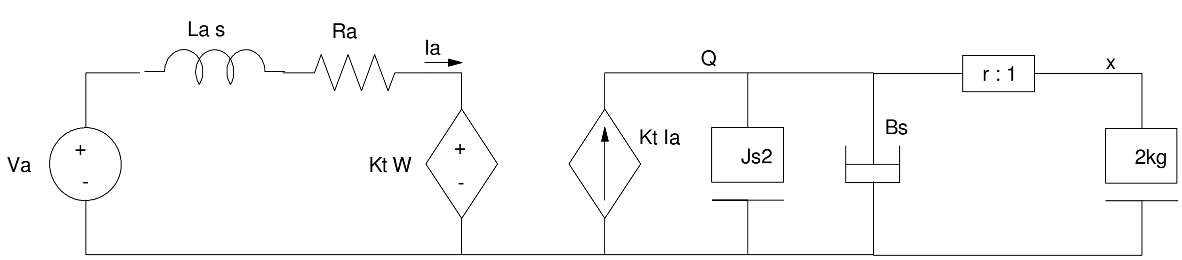
- x is the displacement in meters
- r is the radius of the wheel and
- θ is the angle in radians.

Example: Find the dynamics of the previous motor if it is driving a cart with a mass of 2kg through a wheel with a diameter of 3cm (0.03m)



A wheel is a gear which transates rotation to displacement

First, draw the circuit equivalent



Circuit equivalent for a DC servo motor connected to a wheel and a 2kg cart

Remove the gear by translating the 2kg mass to the motor angle

$$x = r\theta$$

$$2kg \cdot \left(\frac{r}{1}\right)^2 = 0.0018\ kg\ m^2$$

The net inertia is then

$$J_{motor} + J_{mass} = 5.76 \cdot 10^{-6} + 0.0018 = 0.00180576$$

The motor dynamics are then

$$\theta = \left(\frac{13061}{s(s+3.273)(s+387)}\right) V_a$$

The motor dynamics have a complex pole - which might seem odd. Motors are intended to be used to drive something, however. If you drive a cart with a mass of 2kg through wheels with a radius of 3cm, the complex poles shift to { -3.273, -387 }. The complex poles will shift (and probably become real) when connected to a load.