

LaPlace Transforms

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer functions and back.

Example 1: Find the transfer function that corresponds to the following differential equation:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3Y + 6s^2Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

The transfer function from X to Y is

$$G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)$$

Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.

Example 2: Given the transfer function G(s),

$$G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)$$

determine the differential equation relating X and Y.

Solution: Go backwards. X and Y are related by:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

Cross multiply:

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

Note that 'sY' means 'the derivative of Y'

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Sidelight: Fractional powers are not allowed in transfer functions.

- $s^2 Y$ means 'the second derivative of Y'.
- $s^{2.3} Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find $y(t)$ given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

and

$$x(t) = 3 \cos(4t)$$

Solution: This is a phasor problem (Circuits I). $G(s)$ is true for all 's'. For a particular input, all you care about is the specific value of 's' that corresponds to $x(t)$. In this case

$$X = 3 + j0 \quad \text{real} = \text{cosine, imag} = - \text{sine}$$

$$s = j4 \quad \text{the frequency of oscillation}$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j4} (3 + j0)$$

$$Y = -0.5435 - j0.9459$$

meaning

$$y(t) = -0.5435 \cos(4t) + 0.9459 \sin(4t)$$

Note: in phasors,

- real = cosine
- imag = - sine

Example 4: Find $y(t)$ if

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

$$x(t) = 5 \sin(20t)$$

Solution: Similar to before

$$X = 0 - j5$$

$$s = j20$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j20} (0 - j5)$$

$$Y = -0.0230 + j0.0957$$

meaning

$$y(t) = -0.230 \cos(20t) - 0.0957 \sin(20t)$$

Note that the gain varies with frequency (i.e. this is a filter).

Example 5: Find $y(t)$ if

$$x(t) = 3 \cos(4t) + 5 \sin(20t)$$

Solution: Treat this as two separate problems

- $x(t) = 3 \cos(4t)$
- $x(t) = 5 \sin(20t)$

The total input is the sum of the two $x(t)$'s. The total output is the sum of the two $y(t)$'s

$$y(t) = -0.5435 \cos(4t) + 0.9459 \sin(4t) \\ -0.230 \cos(20t) - 0.0957 \sin(20t)$$

Solving Transfer Functions with Step Inputs

There are several ways to do this. My preference is to use a table

Common LaPlace Transforms		
Name	Time: $y(t)$	LaPlace: $Y(s)$
delta (impulse)	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
	$a \cdot e^{-bt}u(t)$	$\frac{a}{s+b}$
	$2a \cdot e^{-bt} \cos(ct - \theta)u(t)$	$\left(\frac{a \angle \theta}{s+b+jc}\right) + \left(\frac{a \angle -\theta}{s+b-jc}\right)$

Example: Find the impulse response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)X$$

$$Y = \left(\frac{5}{s+3}\right)(1)$$

From the above table:

$$y(t) = 5e^{-3t}u(t)$$

Example: Find the step response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)\left(\frac{1}{s}\right) = \left(\frac{5}{s(s+3)}\right)$$

Here we have a problem: the above table doesn't include this type of function. Using partial fractions, however, you can turn it into something that is in the table

$$\left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

Near $s = 0$, both sides go to infinity. The 'B' term is insignificant near $s=0$, so ignore it. Then

$$\left(\frac{5}{s(s+3)}\right)_{s \rightarrow 0} = \left(\frac{A}{s}\right)_{s \rightarrow 0}$$

Clearing out the 's' terms

$$\left(\frac{5}{s+3}\right)_{s \rightarrow 0} = A$$

$$A = 5/3$$

When s approaches -3 , both sides again go to infinity. The 'A' term is finite and insignificant in this case, meaning

$$\left(\frac{5}{s(s+3)}\right)_{s \rightarrow -3} = \left(\frac{B}{s+3}\right)_{s \rightarrow -3}$$

Cancelling the $(s+3)$ terms

$$\left(\frac{5}{s}\right)_{s \rightarrow -3} = B$$

$$B = -5/3$$

So,

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{5/3}{s}\right) - \left(\frac{5/3}{s+3}\right)$$

Using the above table for each term:

$$y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t}\right)u(t)$$

Solving with Repeated Roots: Repeated roots only exist on paper: the probability of a repeated root in practice is zero. For example, the following differential equation has repeated roots:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = x$$

$$(s^2 + 2s + 1)Y = X$$

$$Y = \left(\frac{1}{s^2 + 2s + 1}\right)X$$

$$Y = \left(\frac{1}{(s+1)^2}\right)X$$

The following differential equation does *not* have repeated roots

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 0.9999y = x$$

$$Y = \left(\frac{1}{s^2 + 2s + 0.9999}\right)X$$

$$Y = \left(\frac{1}{(s+1.01)(s+0.99)}\right)X$$

A differential equation is a mathematical model which describes a system. No model is perfect: if the model you're using has repeated roots, perturb it slightly to remove the repeated roots. The response won't be that different.

Solving with Complex Roots:

If you don't mind complex numbers, complex roots are no different than real roots: you just wind up with complex numbers in the partial fraction expansion.

The relevant entry in the LaPlace Transform table is

$$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a \cdot e^{-bt} \cos(ct - \theta)u(t)$$

Example: Find the $y(t)$ given that

$$Y(s) = G \cdot U = \left(\frac{15}{s^2+2s+10}\right) \cdot \left(\frac{1}{s}\right)$$

Solution: Factoring $Y(s)$

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle-161.56^\circ}{s+1+j3}\right) + \left(\frac{0.7906\angle161.56^\circ}{s+1-j3}\right)$$

$$y(t) = 1.5 + 1.5812 \cdot e^{-t} \cdot \cos(3t + 161.56^\circ) \quad \text{for } t > 0$$

note: Control systems is actually easier than signals and systems. In controls, we limit ourselves to one dimensional causal systems (time is one dimensional and only moves forward). For motors, lights, etc. this is a physical limitation. (if you could build a system where the output happens before the input, you can make lots of money in Las Vegas.)

In Signals and Systems, you *can* have non-causal systems. For example, if you are filtering an image, you can filter left to right, right to left, or even up and down. Signals and Systems likewise uses mathematics that work for multiple non-causal dimensions.

Matlab to the Rescue!

Calculating step responses is really tedious. Fortunately, there's Matlab. With it, you can combine systems, find step response, create feedback loops, etc.

Input a system:

Matlab Command

```
G = tf([2,3,4], [5,6,7,8])
```

```
G = zpk([-1,-2], [-3,-4,-5], 10)
```

```
G = ss(A,B,C,D)
```

Meaning

$$G(s) = \left(\frac{2s^2+3s+4}{5s^3+6s^2+7s+8} \right)$$

$$G(s) = \left(\frac{10(s+1)(s+2)}{(s+3)(s+4)(s+5)} \right)$$

$$sX = AX + BU$$

$$Y = CX + DU$$

Step and impulse response

```
t = [0:0.01:10]';
y = impulse(G,t);
plot(t,y)
```

impulse response

```
t = [0:0.01:10]';
y = step(G,t);
plot(t,y)
```

step response

Combining Systems

```
A = zpk([], [-1,-2], 10)
```

$$A(s) = \left(\frac{10}{(s+1)(s+2)} \right)$$

```
B = tf(10, [1,6,4])
```

$$B(s) = \left(\frac{10}{s^2+6s+4} \right)$$

```
G = minreal(A*B)
```

$$G = AB = \left(\frac{10}{(s+1)(s+2)} \right) \left(\frac{10}{s^2+6s+4} \right)$$

```
G = minreal(A + B)
```

$$G = A + B = \left(\frac{10}{(s+1)(s+2)} \right) + \left(\frac{10}{s^2+6s+4} \right)$$

```
G = minreal(A / (1+A*B))
```

$$G = \left(\frac{A}{1+AB} \right) = \left(\frac{\left(\frac{10}{(s+1)(s+2)} \right)}{1 + \left(\frac{10}{(s+1)(s+2)} \right) \left(\frac{10}{s^2+6s+4} \right)} \right)$$