

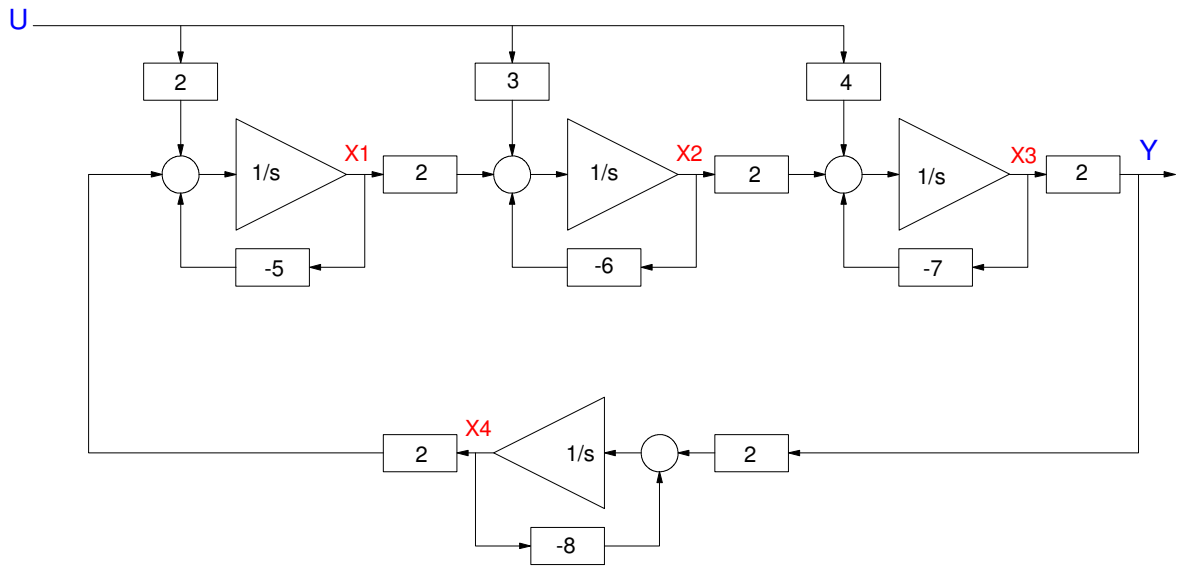
Homework #3: ECE 461 / 661

State-Space, Canonical Forms, RLC Circuits

State-Space

1) Express the following system in state-space form

- Find the transfer function from U to Y



Equations:

$$sX_1 = 2U - 5X_1 + 2X_4$$

$$sX_2 = 3U + 2X_1 - 6X_2$$

$$sX_3 = 4U + 2X_2 - 7X_3$$

$$sX_4 = 4X_3 - 8X_4$$

$$Y = 2X_3$$

In State-Space form

$$s \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 2 \\ 2 & -6 & 0 & 0 \\ 0 & 2 & -7 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix} \bar{X}$$

Find the transfer function using Matlab

```
>> A = [-5,0,0,2 ; 2,-6,0,0 ; 0,2,-7,0 ; 0,0,4,-8]
```

```
    -5     0     0     2  
     2    -6     0     0  
     0     2    -7     0  
     0     0     4    -8
```

```
>> B = [2;3;4;0]
```

```
    2  
    3  
    4  
    0
```

```
>> C = [0,0,2,0]
```

```
    0     0     2     0
```

```
>> D = 0;
```

```
>> G = ss(A,B,C,D);
```

```
>> zpk(G)
```

```
      8 (s+8) (s^2 + 12.5s + 39.5)  
-----  
(s+3.855) (s+9.145) (s^2 + 13s + 46.74)
```

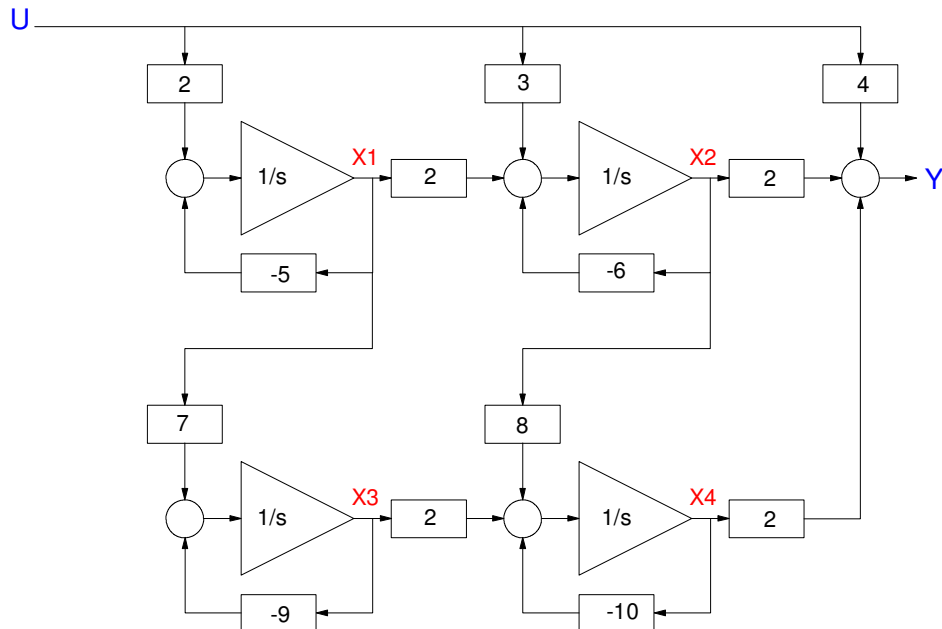
```
>> tf(G)
```

```
      8 s^3 + 164 s^2 + 1116 s + 2528  
-----  
s^4 + 26 s^3 + 251 s^2 + 1066 s + 1648
```

```
>>
```

2) Express the following system in state-space form

- Find the transfer function from U to Y



The equations that describe this system:

$$sX_1 = 2U - 5X_1$$

$$sX_2 = 3U + 2X_1 - 6X_2$$

$$sX_3 = 7X_1 - 9X_3$$

$$sX_4 = 8X_2 + 2X_3 - 10X_4$$

$$Y = 4U + 2X_2 + 2X_4$$

In state-space form

$$s \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 2 & -6 & 0 & 0 \\ 7 & 0 & -9 & 0 \\ 0 & 8 & 2 & -10 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 0 & 2 & 0 & 2 \end{bmatrix} \bar{X} + [4]U$$

Find the transfer function using Matlab

```
>> A = [-5,0,0,0 ; 2,-6,0,0 ; 7,0,-8,0 ; 0,8,2,-10]
```

```
    -5     0     0     0
     2    -6     0     0
     7     0    -8     0
     0     8     2    -10
```

```
>> B = [2;3;0;0]
```

```
    2
    3
    0
    0
```

```
>> C = [0,2,0,2]
```

```
    0     2     0     2
```

```
>> D = 4;
```

```
>> G = ss(A, B, C, D);
```

```
>> zpk(G)
```

```
4 (s+9.505) (s+6.28) (s^2 + 14.72s + 64.53)
-----
(s+10) (s+8) (s+6) (s+5)
```

```
>> tf(G)
```

```
4 s^4 + 122 s^3 + 1426 s^2 + 7588 s + 1.541e004
-----
s^4 + 29 s^3 + 308 s^2 + 1420 s + 2400
```

```
>>
```

Canonical Forms

3) Express the following system in controller canonical form

$$Y = \left(\frac{5(s+2)}{(s+3)(s+4)(s+5)} \right) X$$

In Matlab

```
>> poly([-3,-4,-5])
```

```
1      12      47      60
```

```
>> A = [0,1,0 ; 0,0,1 ; -60, -47, -12]
```

```
0      1      0
0      0      1
-60    -47    -12
```

```
>> B = [0;0;1]
```

```
0
0
1
```

```
>> C = [10, 5, 0]
```

```
10      5      0
```

```
>> D = 0;
```

```
>> G = ss(A, B, C, D);
```

```
>> zpk(G)
```

```
5 (s+2)
-----
(s+5) (s+4) (s+3)
```

4) Express the following system in Jordan (diagonal) canonical form

$$Y = \left(\frac{5(s+2)}{(s+3)(s+4)(s+5)} \right) X$$

Use partial fraction expansion

$$Y = \left(\left(\frac{a}{s+3} \right) + \left(\frac{b}{s+4} \right) + \left(\frac{c}{s+5} \right) \right) X$$

In Matlab

```
>> s = -3;
>> a = 5*(s+2) / ( (s+4)*(s+5) )

a =    -2.5000

>> s = -4;
>> b = 5*(s+2) / ( (s+3)*(s+5) )

b =     10

>> s = -5;
>> c = 5*(s+2) / ( (s+3)*(s+4) )

c =    -7.5000
```

so

$$Y = \left(\left(\frac{-2.5}{s+3} \right) + \left(\frac{10}{s+4} \right) + \left(\frac{-7.5}{s+5} \right) \right) X$$

so

```
>> A = diag([-3,-4,-5])

    -3     0     0
     0    -4     0
     0     0    -5

>> B = [1;1;1]

     1
     1
     1

>> C = [-2.5,10,-7.5]

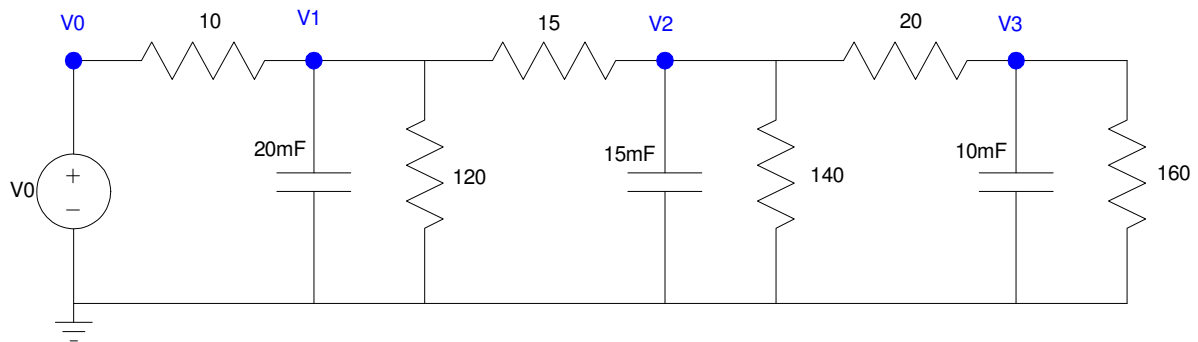
    -2.5000    10.0000   -7.5000

>> D = 0;
>> G = ss(A, B, C, D);
>> zpk(G)

      5 (s+2)
-----
(s+5) (s+4) (s+3)
```

RC Circuits

For the following RC circuit



5) Find the differential equations that describe this circuit

$$0.02sV_1 = \left(\frac{V_0 - V_1}{10} \right) + \left(\frac{0 - V_1}{120} \right) + \left(\frac{V_2 - V_1}{15} \right)$$

$$0.015sV_2 = \left(\frac{V_1 - V_2}{15} \right) + \left(\frac{0 - V_2}{140} \right) + \left(\frac{V_3 - V_2}{20} \right)$$

$$0.01sV_3 = \left(\frac{V_2 - V_3}{20} \right) + \left(\frac{0 - V_3}{160} \right)$$

6) Find the transfer function from V_0 to V_3

Simplifying

$$sV_1 = 5V_0 - 8.75V_1 + 3.333V_2$$

$$sV_2 = 4.444V_1 - 8.254V_2 + 3.333V_3$$

$$sV_3 = 5V_2 - 5.6250V_3$$

Place in matrix form

$$s \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -8.75 & 3.333 & 0 \\ 4.444 & -8.254 & 3.333 \\ 0 & 5 & -5.6250 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} V_0$$

$$Y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Find the transfer function using Matlab

```
>> A = [-8.75 , 3.333 , 0 ; 4.444 , -8.254, 3.333 ; 0, 5, -5.6250]
```

```

-8.7500    3.3330    0
 4.4440   -8.2540    3.3330
 0         5.0000   -5.6250
```

```
>> B = [5;0;0]
```

```

5
0
0
```

```
>> C = [0,0,1];
```

```
>> D = 0;
```

```
>> G = ss(A,B,C,D);
```

```
>> zpk(G)
```

```
111.1
```

```
-----
(s+13.49) (s+7.349) (s+1.786)
```

```
>>
```

7) Assume $V_0 = 0V$ and $V_3(0) = 1V$. What initial condition on V_1 and V_2 makes the energy in the system

- Decay as fast as possible?
- Decay as slow as possible?

This is an eigenvalue / eigenvector problem

```
>> [M,V] = eig(A)

      fast      slow
0.5101    0.6128    0.2798
-0.7260    0.2576    0.5847
0.4613   -0.7471    0.7615

-13.4940   -7.3489   -1.7860

-13.4940         0         0
      0   -7.3489         0
      0         0   -1.7860
```

The initial condition that decays fast is

```
>> M(:,1) / M(3,1)

V1    1.1057
V2   -1.5738
V3    1.0000
```

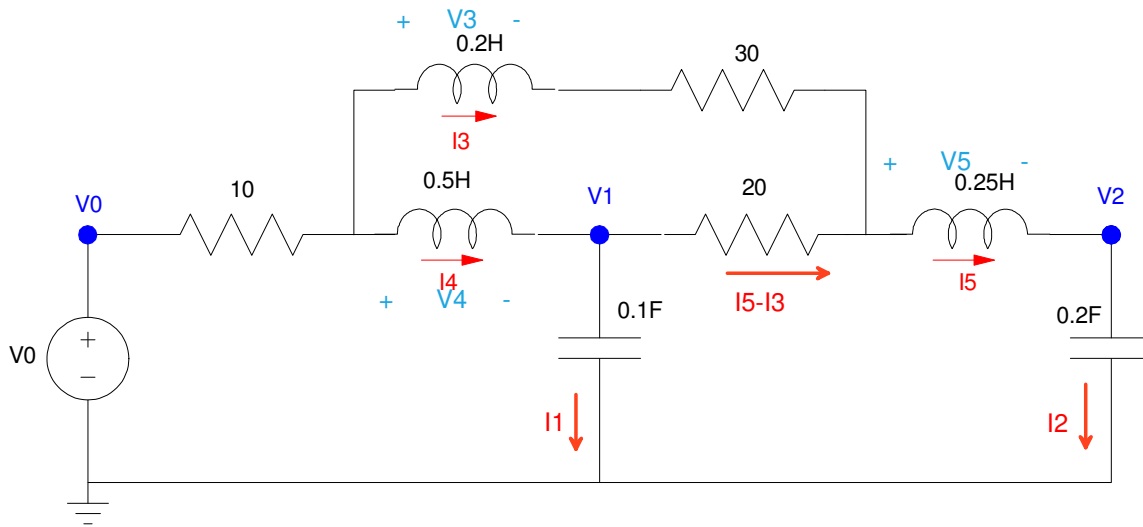
The initial condition that decays slow is

```
>> M(:,3) / M(3,3)

V1    0.3675
V2    0.7678
V3    1.0000
```

RLC Circuits

For the following RLC circuit



8) Find the differential equations that describe this circuit

$$I_1 = 0.1sV_1 = I_4 - (I_5 - I_3)$$

$$I_2 = 0.2sV_2 = I_5$$

$$V_3 = 0.2sI_3 = V_0 - 10(I_3 + I_4) - 30I_3 + 20(I_5 - I_3) - V_1$$

$$V_4 = 0.5sI_4 = V_0 - 10(I_3 + I_4) - V_1$$

$$V_5 = 0.25sI_5 = V_1 - 20(I_5 - I_3) - V_2$$

9) Find the transfer function from V_0 to V_2

Group terms

$$sV_1 = 10I_3 + 10I_4 - 10I_5$$

$$sV_2 = 5I_5$$

$$sI_3 = 5V_0 - 5V_1 - 300I_3 - 50I_4 + 100I_5$$

$$sI_4 = 2V_0 - 2V_1 - 20I_3 - 20I_4$$

$$sI_5 = 4V_1 - 4V_2 + 80I_3 - 80I_5$$

Place in matrix form

$$s \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 10 & 10 & -10 \\ 0 & 0 & 0 & 0 & 5 \\ -5 & 0 & -300 & -50 & 100 \\ -2 & 0 & -20 & -20 & 0 \\ 4 & -4 & 80 & 0 & -80 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \\ 0 \end{bmatrix} V_0$$

$$Y = V_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix}$$

Solve using Matlab

```
>> a1 = [0,0,10,10,-10];
>> a2 = [0,0,0,0,5];
>> a3 = [-5,0,-300,-50,100];
>> a4 = [-2,0,-20,-20,0];
>> a5 = [4,-4,80,0,-80];
>> A = [a1;a2;a3;a4;a5]
```

```
    0    0    10    10   -10
    0    0     0     0     5
   -5    0  -300   -50   100
   -2    0   -20   -20     0
    4   -4    80     0   -80
```

```
>> B = [0;0;5;2;0]
```

```
    0
    0
    5
    2
    0
```

```
>> C = [0,1,0,0,0];
>> D = 0;
>> G = ss(A,B,C,D);
```

```
>> zpk(G)
```

$$\frac{2000 (s^2 + 0.7s + 50)}{(s+334.3) (s+51.02) (s+12.05) (s+2.378) (s+0.2046)}$$

```
>> tf(G)
```

$$\frac{2000 s^2 + 1400 s + 1e005}{s^5 + 400 s^4 + 2.273e004 s^3 + 2.618e005 s^2 + 5.414e005 s + 1e005}$$

```
>>
```

10) Assume $V_0(t) = 0$ and $V_2(0) = 1V$. What initial condition on $V_1(0)$, $I_3(0)$, and $I_4(0)$ results in the energy in the system

- Decaying as fast as possible?
- Decaying as slow as possible?

This is an eigenvalue / eigenvector problem. The eigenvectors and eigenvalues are

```
>> [M,V] = eig(A)

    0.0392    -0.0742    -0.6311     0.9505    -0.4537
   -0.0045     0.0896     0.1195    -0.2345    -0.8897
   -0.9516    -0.3231    -0.2070     0.0490     0.0145
   -0.0603    -0.2131     0.6795    -0.1634     0.0312
    0.2986    -0.9147    -0.2879     0.1115     0.0364

  -334.3435  -51.0246  -12.0494   -2.3778   -0.2046
```

The fast mode (decays as $\exp(-334t)$) is:

```
>> M(:,1) / M(2,1)

V1    -8.7770
V2     1.0000
I3   213.0841
I4    13.5016
I5   -66.8687
```

The slow mode (decays as $\exp(-0.2046t)$) is:

```
>> M(:,5) / M(2,5)

V1     0.5099
V2     1.0000
I3    -0.0163
I4    -0.0350
I5    -0.0409

>>
```