Filters in the z-Plane

NDSU ECE 376

Lecture #28

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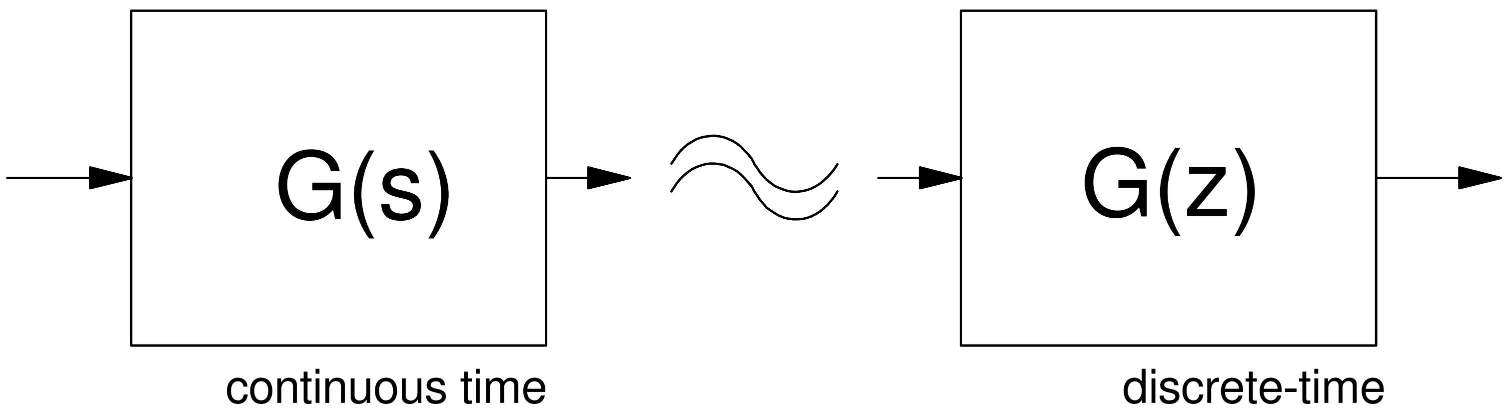
Please visit Bison Academy for corresponding lecture notes, homework sets, and solutions

Filters in the z-Plane

Given G(s), find G(z) so that

- They both have the same step response, or
- They both have the same frequency response.



LaPlace Operator (s)

LaPlace Transforms assume

$$y(t) = e^{st}$$

giving

$$\frac{dy}{dt} = s \cdot e^{st} = sy$$

$$sY \textit{ means 'the derivative of } y(t)\textit{'}$$

A transfer function, such as

$$Y = \left(\frac{2s+3}{s^2+2s+10}\right)X$$

is equivalent to

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 2\frac{dx}{dt} + 3x$$

z-Operator

z-Transforms assume

$$y(k) = z^k$$

giving

$$y(k+1) = z^{k+1} = z \cdot z^k = zy(k)$$

zY means 'the next value of y(k)

A transfer function, such as

$$Y = \left(\frac{2z+3}{z^2+2z+10}\right)X$$

means

```
y(k+2) + 2y(k+1) + 10y(k) = 2x(k+1) + 3x(k)
```

s to z-Plane Relationship

Assume

$$t = kT$$

where

- k is the sample number
- T is the sampling time (one sample every T seconds)

Substituting

$$y(kT) = e^{skT} = (e^{sT})^k = (z)^k$$

$$\boxed{z = e^{sT}}$$

Filter Analysis in the s-Plane

Find y(t)

$$Y = \left(\frac{2}{s+3}\right) X$$

$$x(t) = 4\cos(5t) + 6\sin(5t)$$

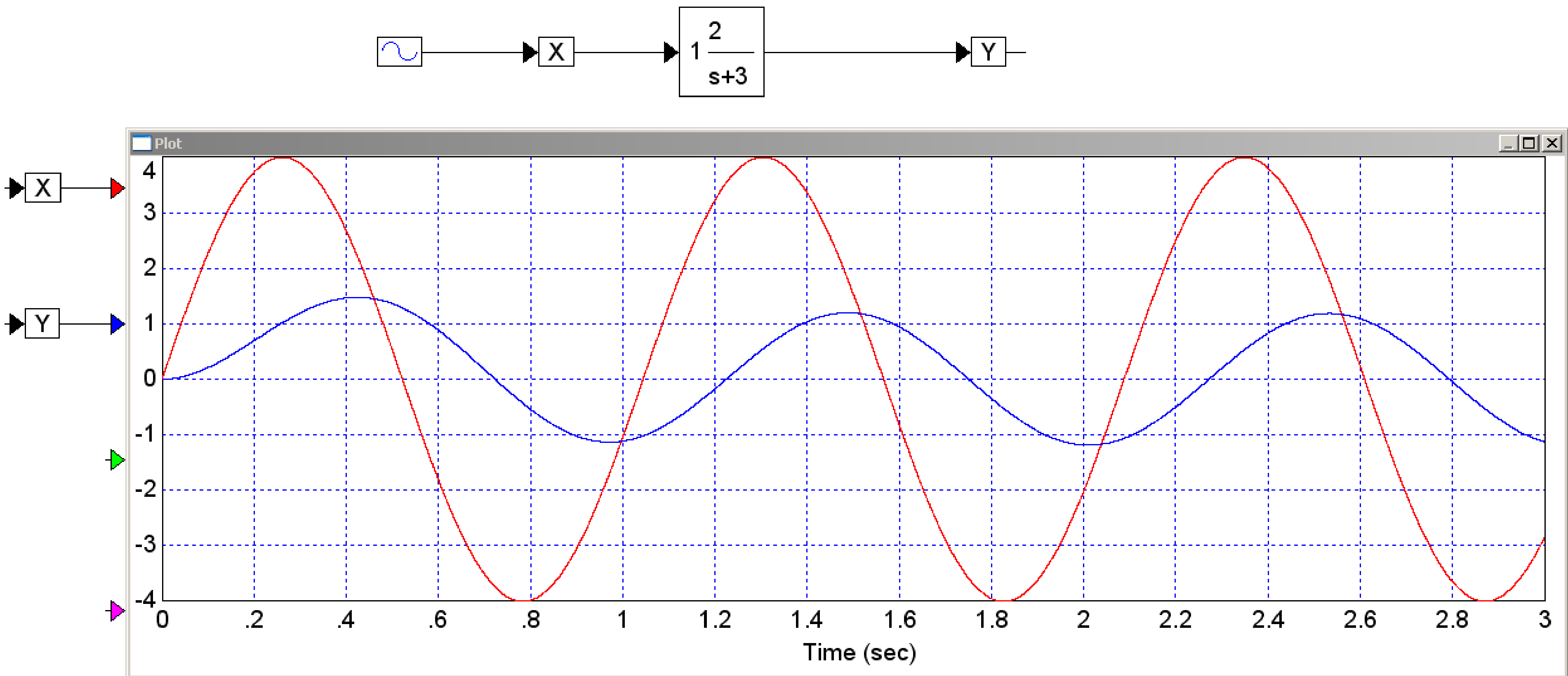
Solution:

$$Y = \left(\frac{2}{s+3}\right)_{s=j5} (4 - j6)$$

$$Y = -1.059 - j2.235$$

$$y(t) = -1.059\cos(5t) + 2.235\sin(5t)$$

s-Plane: Sine-wave input produces a sine-wave output



Filter Analysis in the z-Plane

$$z = e^{sT} = e^{j\omega T}$$

Find y(t). Assume T = 0.01 second

$$Y = \left(\frac{0.2}{z-0.9}\right) X$$

$$x(t) = 4\cos(5t) + 6\sin(5t)$$

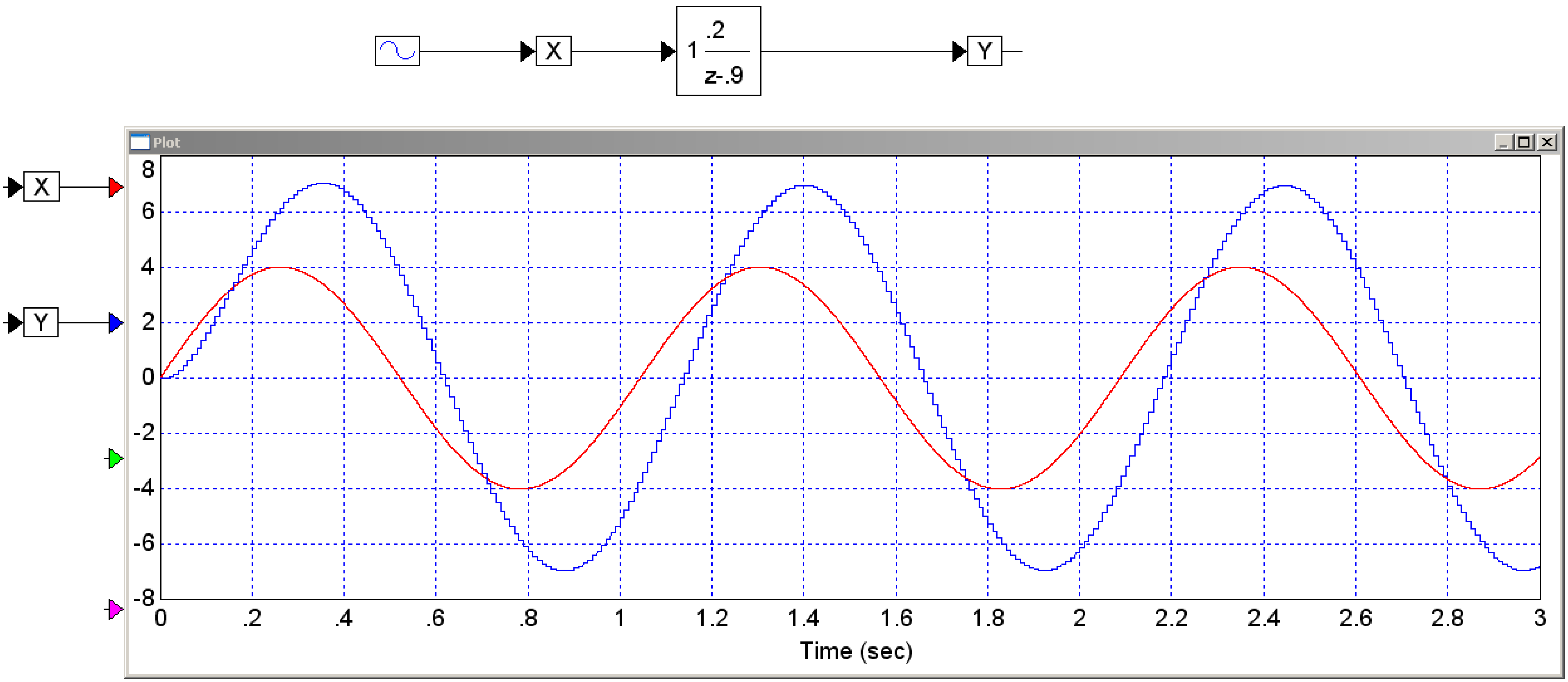
Solution: Determine the gain at $z = e^{j5T}$

$$Y = \left(\frac{0.2}{z-0.9}\right)_{z=e^{j0.05}} (4 - j6)$$

$$Y = -2.575 - j1.548$$

$$y(t) = -2.575\cos(5t) + 1.548\sin(5t)$$

z-Plane: Sine wave input produces a sine-wave output



Converting G(s) to G(z)

- Poles convert as $z = e^{sT}$
- Zeros convert as $z = e^{sT}$
- Add a gain to match the gain at one frequency (typically DC)

Example 1: Convert to the z-plane. Assume T = 0.01

$$G(s) = \left(\frac{30}{(s+2)(s+10)}\right)$$

Solution: Convert the poles to the z-plane

s = -2 $\qquad z = e^{-2T} = 0.9802$

s = -10 $\qquad z = e^{-10T} = 0.9048$

so $\qquad G(z) = \left(\frac{k}{(z-0.9802)(z-0.9048)} \right)$

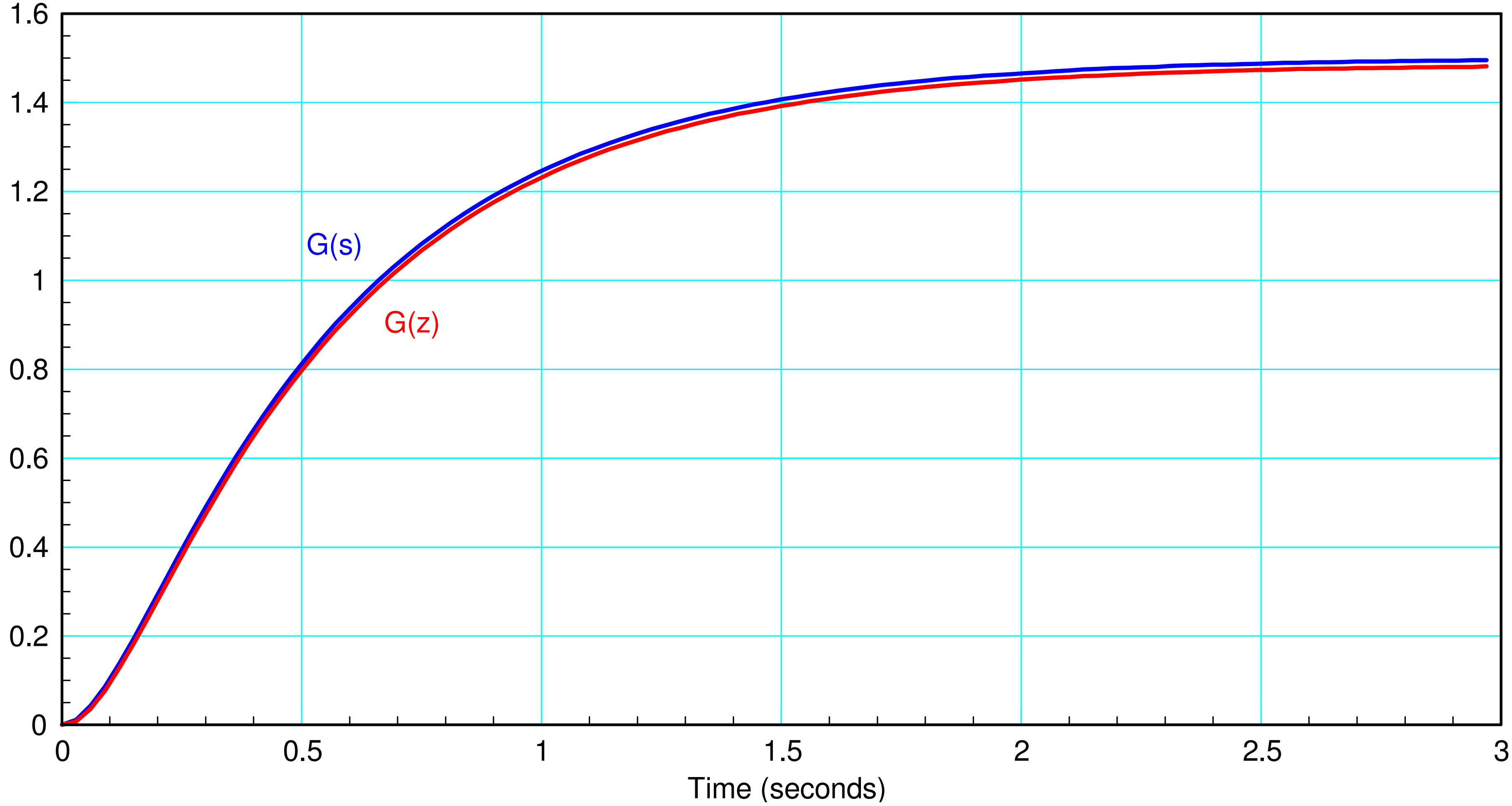
To find k, match the gain at DC

$$\left(\frac{30}{(s+2)(s+10)} \right)_{s=0} = \left(\frac{k}{(z-0.9802)(z-0.9048)} \right)_{z=1} = 1.5$$

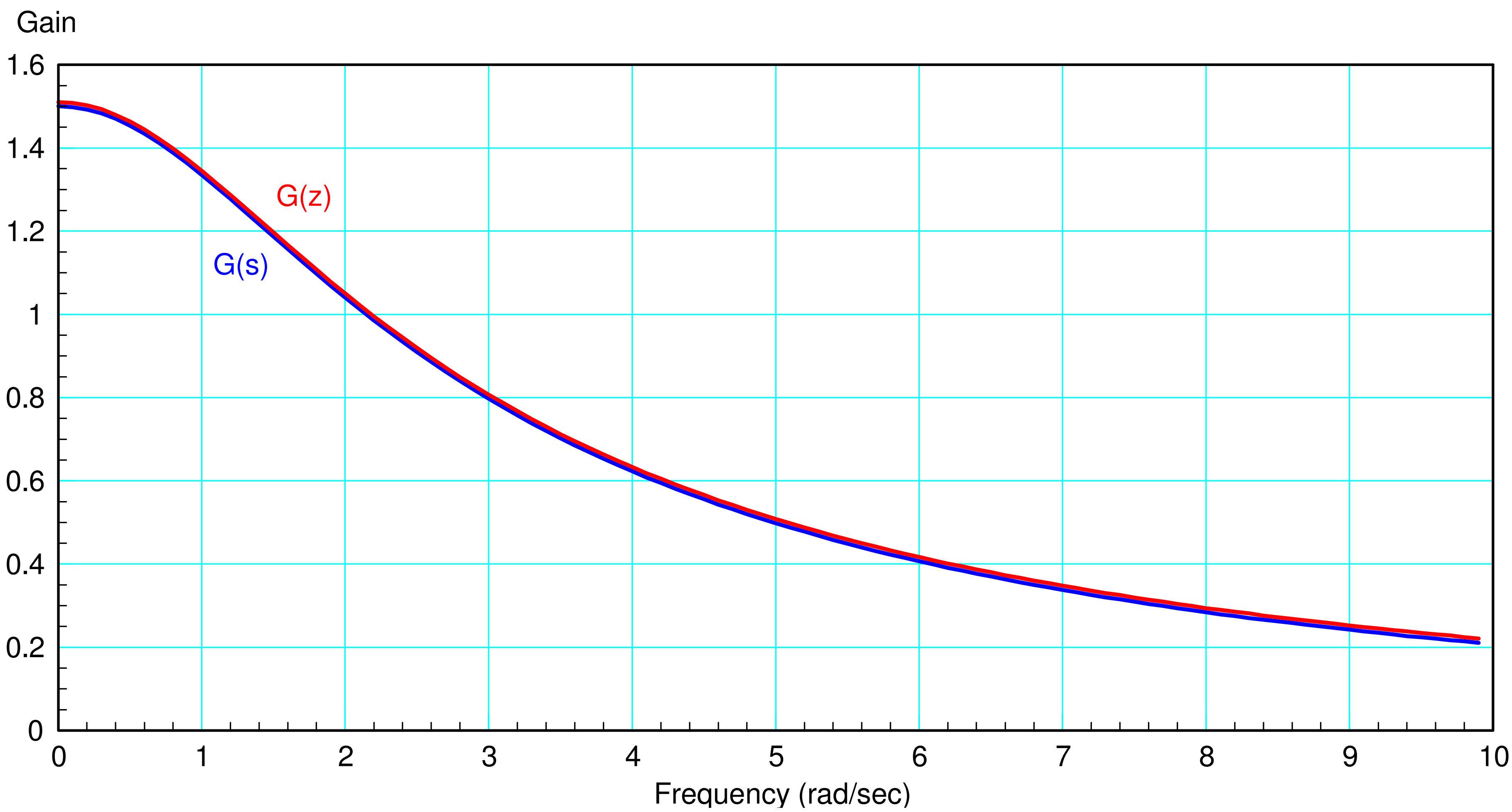
$$k = 0.002827$$

$$\left(\frac{30}{(s+2)(s+10)} \right) \approx \left(\frac{0.002827}{(z-0.9802)(z-0.9048)} \right)$$

G(s) and G(z) have the same step response



G(s) and G(z) have the same frequency response



Implementing G(z)

Write a program to implement

$$Y = \left(\frac{0.2(z-0.9)}{z^3-1.3z^2+1.6z+0.6}\right)X$$

Cross multiply

$$(z^3 - 1.3z^2 + 1.6z + 0.6)Y = 0.2(z - 0.9)X$$

Write the difference equation

```
y(k+3) - 1.3y(k+2) + 1.6y(k+1) + 0.6y(k) =

           0.2( x(k+1) - 0.9x(k) )
```

Solve for the highest value of y(k+2)

```
y(k+3) =  1.3y(k+2) - 1.6y(k+1) - 0.6y(k) +

             0.2( x(k+1) - 0.9x(k) )
```

Time shift (change of variable: k+3 = k')

```
y(k') =  1.3y(k'-1) - 1.6y(k'-2) - 0.6y(k'-3)

      + 0.2( x(k'-2) - 0.9x(k'-3) )
```

This is essentially the program

$$Y = \left(\frac{0.2(z-0.9)}{z^3-1.3z^2+1.6z+0.6} \right) X$$

```
while(1) {
   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y3 = y2;
   y2 = y1;
   y1 = y0;
   y0 = 1.3*y1 - 1.6*y2 - 0.6*y3 + 0.2*( x2 - 0.9*x3 );

   D2A(y0);

   Wait_T();
   }
```

Note

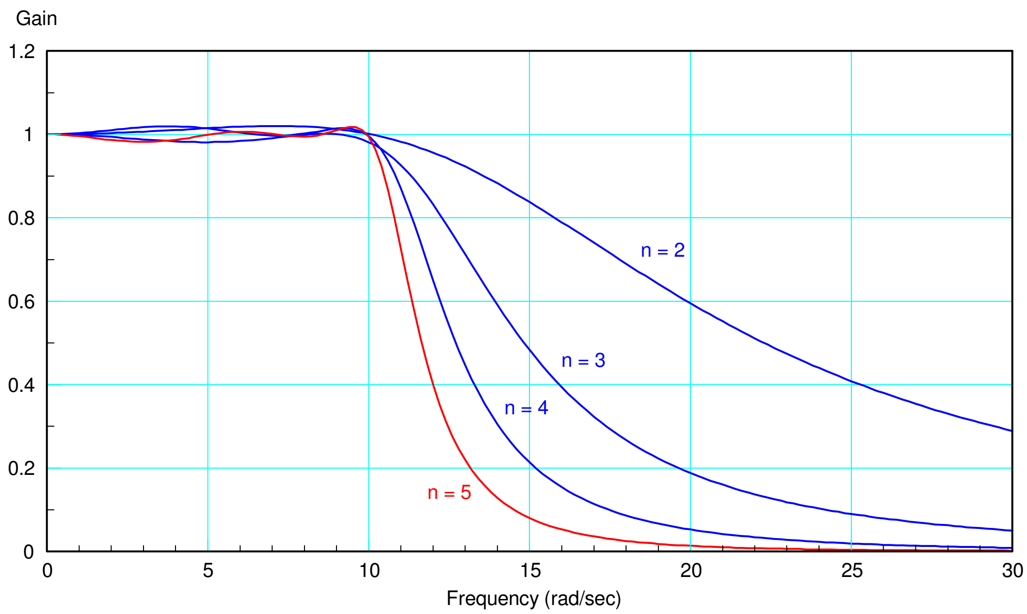
- If you want to change the filter, you just change one line of code
- If you want complex poles or zeros, just choose coefficients that have complex roots

Example 2: Design a digital low-pass filter

$$G(j\omega) \approx \begin{cases} 1 & \omega < 10 \\ 0 & \omega > 10 \end{cases}$$

From lecture #26, a 5th-order Chebychev filter is

$$G(s) = \left(\frac{4.8 \cdot 7.6^2 \cdot 10.6^2}{(s+4.8)\left(s+7.6\angle\pm 59.3^0\right)\left(s+10.6\angle\pm 82^0\right)} \right)$$



Convert G(s) to G(z)

$$G(s) = \left(\frac{4.8 \cdot 7.6^2 \cdot 10.6^2}{(s+4.8)\left(s+7.6\angle\pm59.3^0\right)\left(s+10.6\angle\pm82^0\right)} \right)$$

Assume T = 10ms

Convert using $z = e^{sT}$

$$s = -4.8 \qquad z = 0.9531$$

$$s = -7.6\angle\pm 59.3^0 \qquad z = 0.9599 \pm j0.0626$$

$$s = -10.6\angle\pm 82^0 \qquad z = 0.9799 \pm j0.1032$$

so

$$G(z) = \left(\frac{k}{(z-0.9531)(z-0.9599\pm j0.0626)(z-0.9799\pm j0.1032)} \right)$$

Pick 'k' s that the DC gain is 1.000

$$\left(\frac{k}{(z-0.9531)(z-0.9599\pm j0.0626)(z-0.9799\pm j0.1032)}\right)_{z=1} = 1$$

$$k = 2.8797 \cdot 10^{-6}$$

so

$$G(z) = \left(\frac{2.8797\cdot 10^{-6}}{(z-0.9531)(z-0.9599\pm j0.0626)(z-0.9799\pm j0.1032)}\right)$$

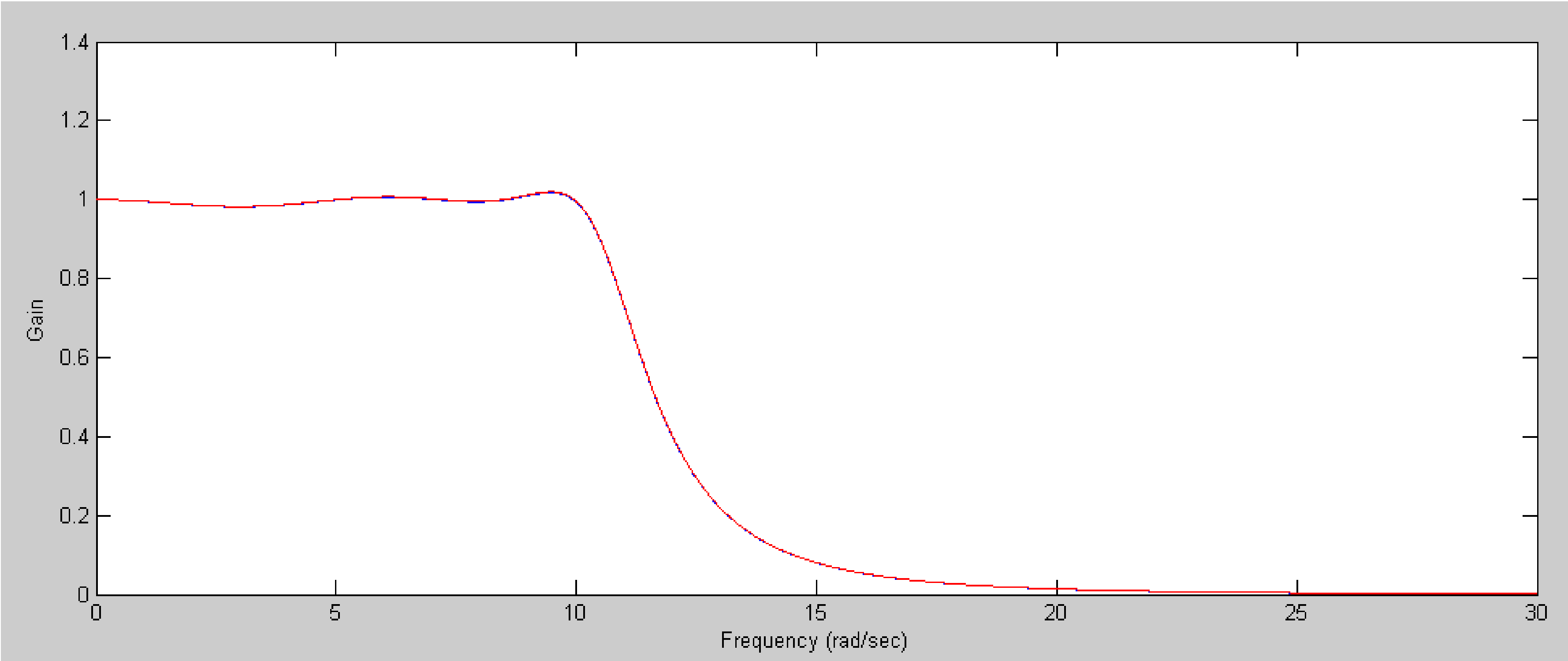
In Matlab:

```
s1 = -4.8;
s2 = -7.6*exp(j*59.3*pi/180);
s3 = conj(s2);
s4 = -10.6*exp(j*82*pi/180);
s5 = conj(s4);
ks = abs( prod([s1,s2,s3,s4,s5]) );

T=0.01;

z1 = exp(s1*T);
z2 = exp(s2*T);
z3 = exp(s3*T);
z4 = exp(s4*T);
z5 = exp(s5*T);
kz = abs(prod([z1-1,z2-1,z3-1,z4-1,z5-1]));
```

```
w = [0:0.01:30]';
s = j*w;
z = exp(s*T);
Gs = ks ./ ( (s-s1).*(s-s2).*(s-s3).*(s-s4).*(s-s5) );
Gz = kz ./ ( (z-z1).*(z-z2).*(z-z3).*(z-z4).*(z-z5) );
plot(w,abs(Gs),'b',w,abs(Gz),'r');
```



Summary:

Converting an analog filter, G(s), to a digital filter, G(z), is fairly easy

- Zeros convert as $z = e^{sT}$
- Poles convert at $z = e^{sT}$
- Pick 'k' to match the DC gain

Once you have the digital filter, it's fairly straight forward to write the corresponding code

- 1st and 2nd-order filters are easier to code and have better numerical properties
- Split up the 5th order filter into cascaded 1st & 2nd order filters

$$G(z) = \left(\frac{0.0469}{z-0.9531}\right)\left(\frac{0.0056}{z-0.9599 \pm j0.0626}\right)\left(\frac{0.0111}{z-0.9799 \pm j0.1032}\right)$$