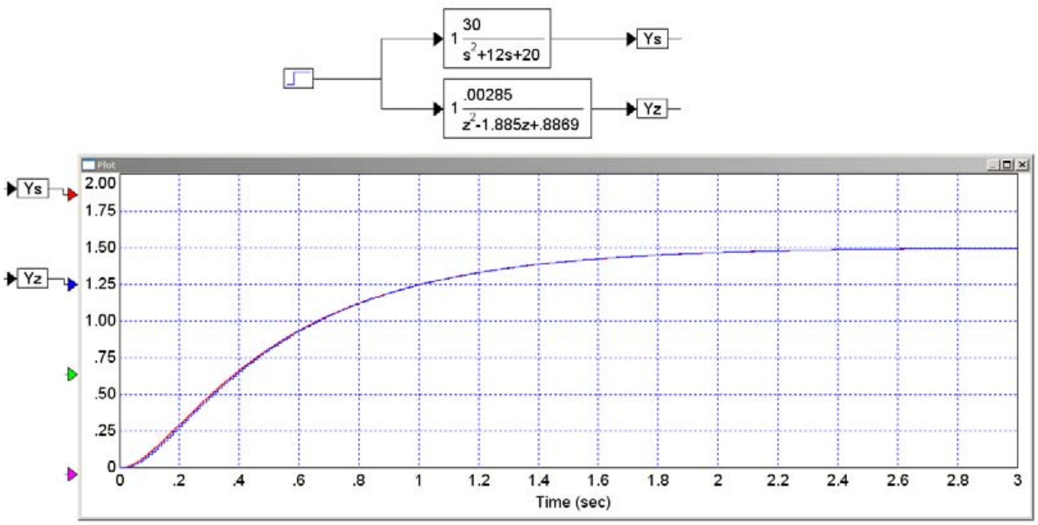
Filters in the z-Plane

Here we look at coming up with a digital filter, G(z), which is equivalent to an anlog filter, G(s). Equivalent meaning

- They both have the same step response, or
- They both have the same frequency response.

For example, the following filters have the same step response. They are equivalent

- One is analog (s-plane)
- One is digital (z-plane)



$$\left(\frac{30}{(s+2)(s+10)}\right) \approx \left(\frac{0.0028}{(z-0.0802)(z-0.9048)}\right)$$

The basic assumption behind LaPlace transforms is that all functions are in the form of

$$y(t) = e^{st}$$

This assumptions turns differential equations into algebraic equations in 's' where sY means 'the derivative of Y'.

$$\frac{dy}{dt} = s \cdot e^{st} = sy$$

A transfer function, such as

$$Y = \left(\frac{2s+3}{s^2+2s+10}\right) X$$

thus represents the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 2\frac{dx}{dt} + 3x$$

The basic assumption behind z-transforms is that all functions are in the form of

$$y(k) = z^k$$

This assumption turns difference equations into algebraic equations in 'z' where zY means 'the next value of Y'

$$y(k+1) = z^{k+1} = z \cdot z^k = zy(k)$$

A transfer function, such as

$$Y = \left(\frac{2z+3}{z^2+2z+10}\right)X$$

this represents a difference equation

y(k+2) + 2y(k+1) + 10y(k) = 2x(k+1) + 3x(k)

The relationship between the s-plane and z-plane is as follows. Assume that time is sampled at a sampling rate of T seconds. Then

$$t = kT$$

where k is the sample number. Plugging this into the LaPlace assumption

$$y(kT) = e^{skT}$$

$$y(k) = (e^{sT})^k$$

or

$$y(k) = z^k$$

Hence, the transformation between the s-plane and the z-plane is

$$\boxed{z = e^{sT}}$$

Gain for a Sinusoidal Input

If filter, G(s), has a sinusoidal input, you can find the gain at that frequency by substituting

$$s \rightarrow j\omega$$

The result is a comples number where

- the amplitude is how much the signal is scaled, and
- the angle is the phase shift

For example, find y(t) assuming

$$Y = \left(\frac{2}{s+3}\right)X$$

$$x(t) = 4\cos(5t)$$

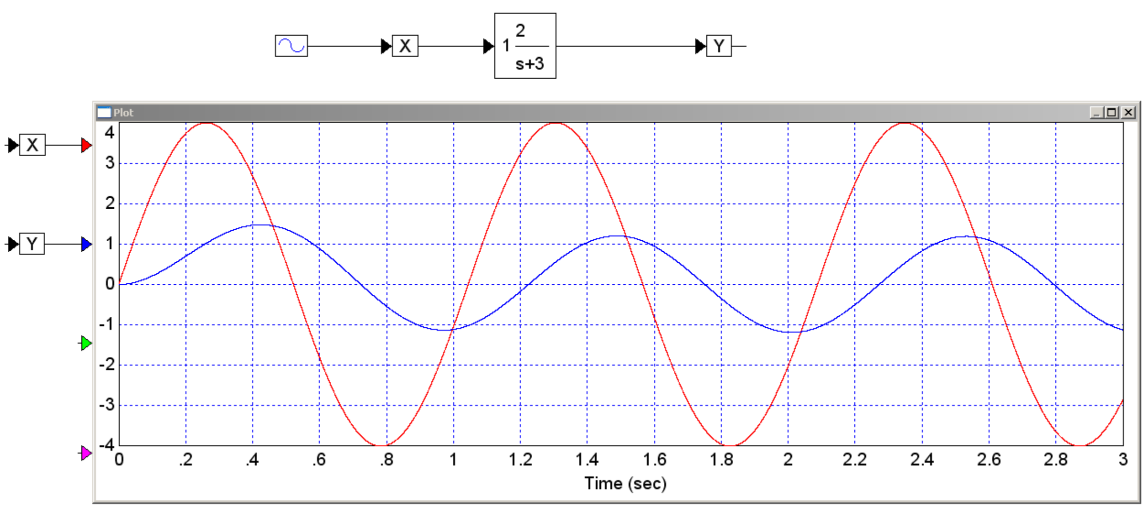
Solution: Determine the gain at s = j5

$$\left(\frac{2}{s+3}\right)_{s=j5} = 0.343\angle -59^0$$

$$y(t) = (0.343\angle -59^0)\cdot 4\cos(5t)$$

so

$$\boxed{y(t) = 1.372\cos(5t - 59^0)}$$



If a filter, G(z), has a sinusoidal input, you can find the gain at that frequency by substituting

$$z = e^{sT} = e^{j\omega T}$$

For example, find y(t) assuming a sampling rate of 0.01 second (T = 0.01)

$$Y = \left(\frac{0.2}{z-0.9}\right)X$$

$$x(t) = 4\cos(5t)$$

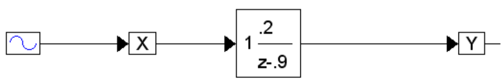
Solution: Determine the gain at $z = e^{j5T}$

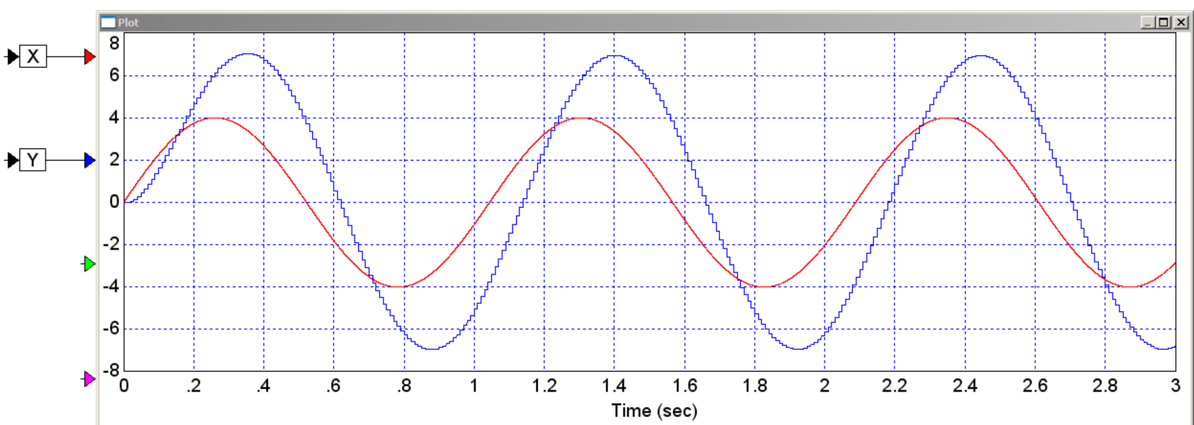
$$\left(\frac{0.2}{z-0.9}\right)_{z=e^{j0.05}} = 1.8071\angle-26.84^0$$

so

$$y(t) = (1.8071\angle-26.84^0)\cdot 4\cos(5t)$$

$$y(t) = 7.2282\cos(5t-26.84^0)$$





Note that if you change the sampling rate, you change z as well as the gain. Digital filters require a fixed known sampling rate.

Converting G(s) to G(z)

You can convert a filter from the s-plane to the z-plane using the relationship $z = e^{sT}$. Given G(s)

- Convert the zeros to the z-plane as $z = e^{sT}$
- Convert the poles to the z-plane as $z = e^{sT}$
- Add a gain to match the gain at one frequency (typically DC)

Example 1: Find a digital filter with approximately the same frequency response as

$$G(s) = \left(\frac{30}{(s+2)(s+10)}\right)$$

Assume a sampling rate of 10ms (T = 0.01)

Solution: Convert the poles to the z-plane

s = -2

$$z = e^{-2T} = 0.9802$$

s = -10

$$z = e^{-10T} = 0.9048$$

so

$$G(z) = \left(\frac{k}{(z-0.9802)(z-0.9048)}\right)$$

To find k, match the gain at DC

$$\left(\frac{30}{(s+2)(s+10)}\right)_{s=0} = 1.5$$

so

$$\left(\frac{k}{(z-0.9802)(z-0.9048)}\right)_{z=1} = 1.5$$

k = 0.0028

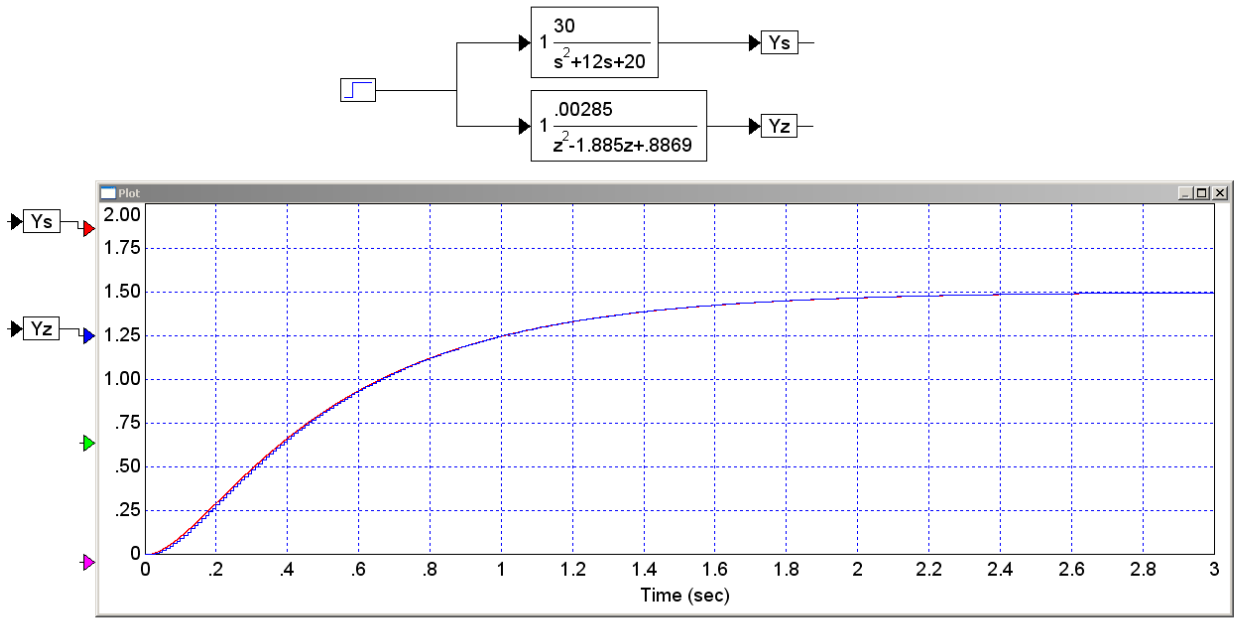
and

$$G(z) = \left(\frac{0.0028}{(z-0.0802)(z-0.9048)}\right)$$

Hence, for T = 0.01

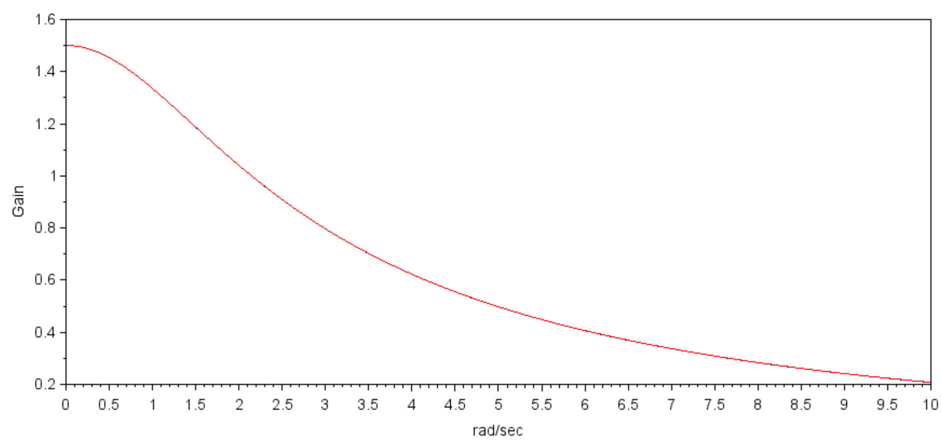
$$\left(\frac{30}{(s+2)(s+10)}\right) \approx \left(\frac{0.0028}{(z-0.0802)(z-0.9048)}\right)$$

Checking: if the two systems are equivalent, they should have the same step response



If the two systems are the same, they should have the same frequency response. In MATLAB:

```
w = [0:0.01:10]';
j = sqrt(-1);
s = j*w;
Gs = 30 . / ( (s+2).*(s+10) );
T = 0.01;
z = exp(s*T);
Gz = 0.00285 ./ ( (z-0.9801).*(z-0.9048) );
plot(w,abs(Gs),w,abs(Gz));
xlabel('rad/sec');
ylabel('Gain');
```



Gain of G(s) (blue) and G(z) (red) (too close to tell apart)

Programming G(z)

Given a filter G(z), a program to impliment it is fairly straight forward:

- First find the difference equation
- Solve for the highest value of y(k+n)
- Time shift it (change of variable) so that the filter is causal

Example: Write a program to impliment

$$Y = \left(\frac{0.2(z-0.9)}{z^3-1.3z^2+1.6z+0.6}\right)X$$

Cross multiply

$$(z^3 - 1.3z^2 + 1.6z + 0.6)Y = 0.2(z-0.9)X$$

Write the difference equation

```
y(k+3) - 1.3y(k+2) + 1.6y(k+1) + 0.6y(k) = 0.2( x(k+1) - 0.9x(k) )
```

Solve for the highest value of y(k+2)

```
y(k+3) =  1.3y(k+2) - 1.6y(k+1) - 0.6y(k) = 0.2( x(k+1) - 0.9x(k) )
```

Time shift (change of variable: k+3 = k')

```
y(k') =  1.3y(k'-1) - 1.6y(k'-2) - 0.6y(k'-3) + 0.2( x(k'-2) - 0.9x(k'-3) )
```

Write a program. For this program, you need to remember the last three inputs and outputs

```
while(1) {
   x3 = x2;
   x2 = x1;
   x1 = x0;
   x0 = A2D_Read(0);

   y3 = y2;
   y2 = y1;
   y1 = y0;
   y0 = 1.3*y1 - 1.6(y2 - 0.6*y3 + 0.2*( x2 - 0.9*x3 );

   D2A(y0);

   Wait_T();       // set the sampling rate, presumably with interrupts
   }
```

Note

- If you want to change the filter, you just change one line of code
- If you want comples poles or zeros, just choose coefficients that have complex roots

Digital filters are much much easier to impliment than analog filters.