Uniform Distribution

ECE 341: Random Processes

Lecture #12

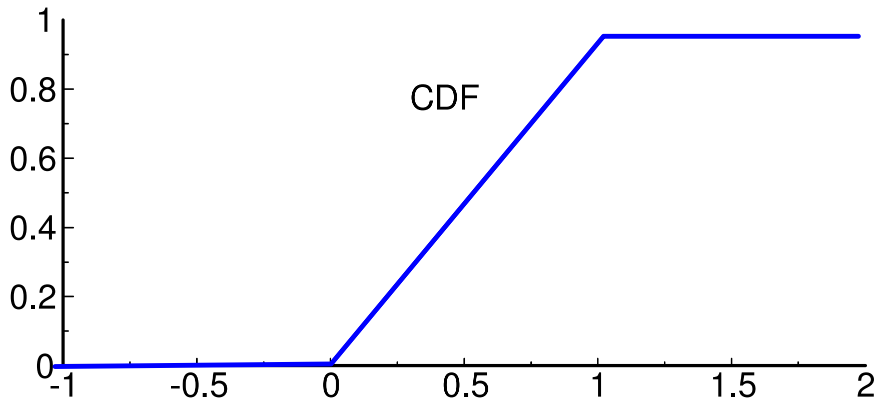
note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

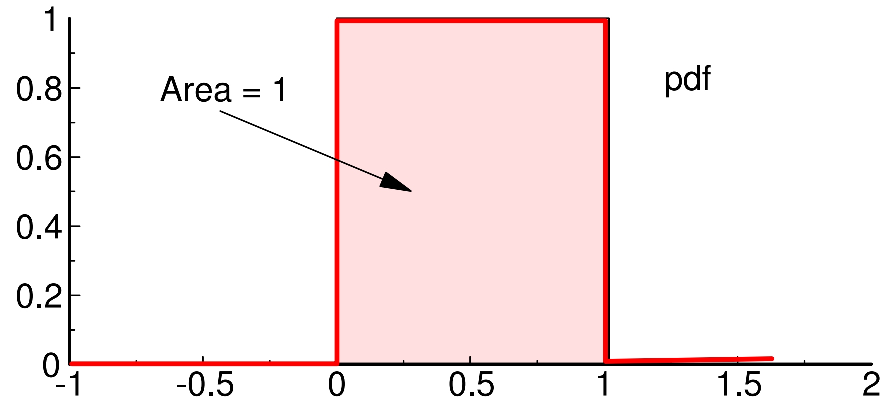
Uniform Distribution

A uniform distribution is one where the probability density function (pdf) is

- constant over a range (a, b), and
- zero otherwise.

Example: Uniform distribution over the interval of (1, 2):





Examples:

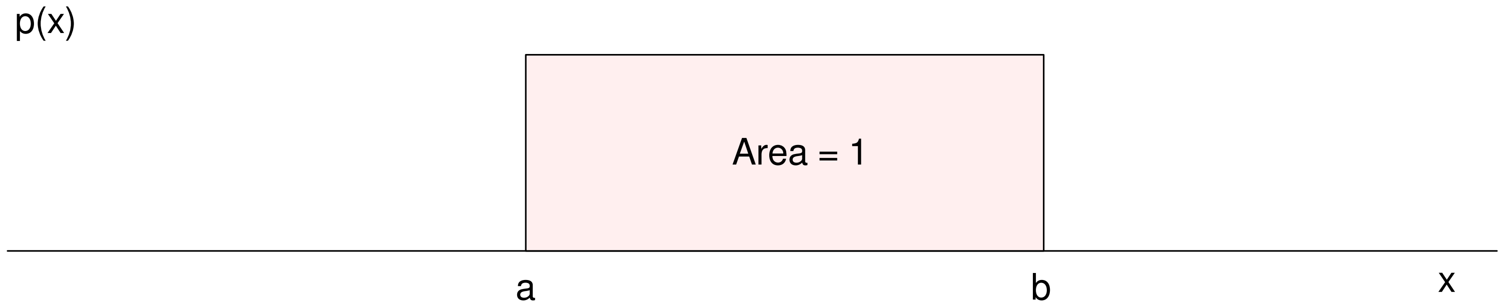
- A 1k resistor with a tolerance of 5% will have a value in the range of +/- 5%. This can be modeled as a uniform distribution over the range of 950 < R < 1050
- A 3904 transistor has a nominal gain of 200 with a variation of +/- 100. This can be modeled as a unifiom distrubtion over the range if 100 < gain < 300

In general, if you know the limits but know nothing else about the distribution, a uniform distribution is a reasonable assumption to make about the random variable.

TYPE NO.	DESCRIPTION	LEAD CODE	V_{CBO} (V) MIN	V_{CEO} (V) $^*V_{CES}$ MIN	V_{EBO} (V) MIN	I_{CBO} @ (nA) $^*I_{CES}$ $^*I_{CEV}$ MAX	V_{CB} (V)	h_{FE} MIN	$^*h_{FE}$ (1kHZ) MAX	@ V_{CE} (V)	@ I_C (mA)	$V_{CE (SAT)}$ (V)	@ I_C (mA)	C_{ob} (pF) $^*C_{rb}$ MAX	f_T (MHz) *TYP MIN	NF (dB) MAX	t_{off} MAX
2N3860	NPN LOW NOISE	ECB	30	30	4.0	50	30	150	300	4.50	2.0	0.125	10	4.0	90	- -	- -
2N3903	NPN AMPL/SWITCH	EBC	60	40	6.0	50*	30	50	150	1.0	10	0.30	50	4.0	250	6.0	225
2N3904	NPN AMPL/SWITCH	EBC	60	40	6.0	50*	30	100	300	1.0	10	0.30	50	4.0	300	5.0	250
2N3905	PNP AMPL/SWITCH	EBC	40	40	5.0	50*	30	50	150	1.0	10	0.40	50	4.5	200	5.0	260
2N3906	PNP AMPL/SWITCH	EBC	40	40	5.0	50*	30	100	300	1.0	10	0.40	50	4.5	250	4.0	300

Properties of Uniform Distributions:

Assume a uniform distribution over the interval (a, b)



Since the area must be 1.000, the height must be

Area = width * height = 1

$p(x) = \left(\frac{1}{b-a}\right)$ $\qquad$ a < x < b

Mean: The mean of the funciton (almost by inspection) is

$$\bar{x} = \left(\frac{a+b}{2}\right)$$

Variance:

$$\sigma^2 = \int_a^b p(x)\ (x-\mu)^2\ dx$$

$$\sigma^2 = \left(\frac{(b-a)^2}{12}\right)$$

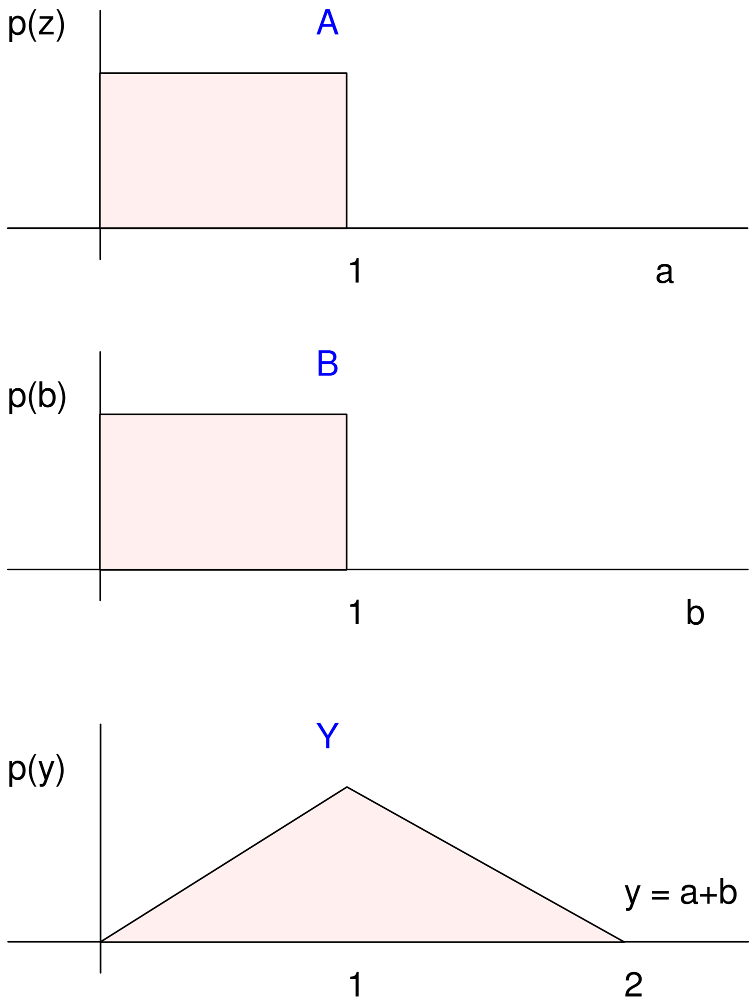
Moment Generating Function

$$\psi(s) = \left(\frac{1}{(b-a)s}\right)(e^{-as} - e^{-bs})$$

Combinations of Uniform Distributions

If you add two uniform distributions, the result is

- The colvolution of their pdf's, or
- The product of their moment generating functions.



Example:

- Assume A and B are uniform distributions over the interval (0, 1).
- Find the pdf of the sum Y = A + B.

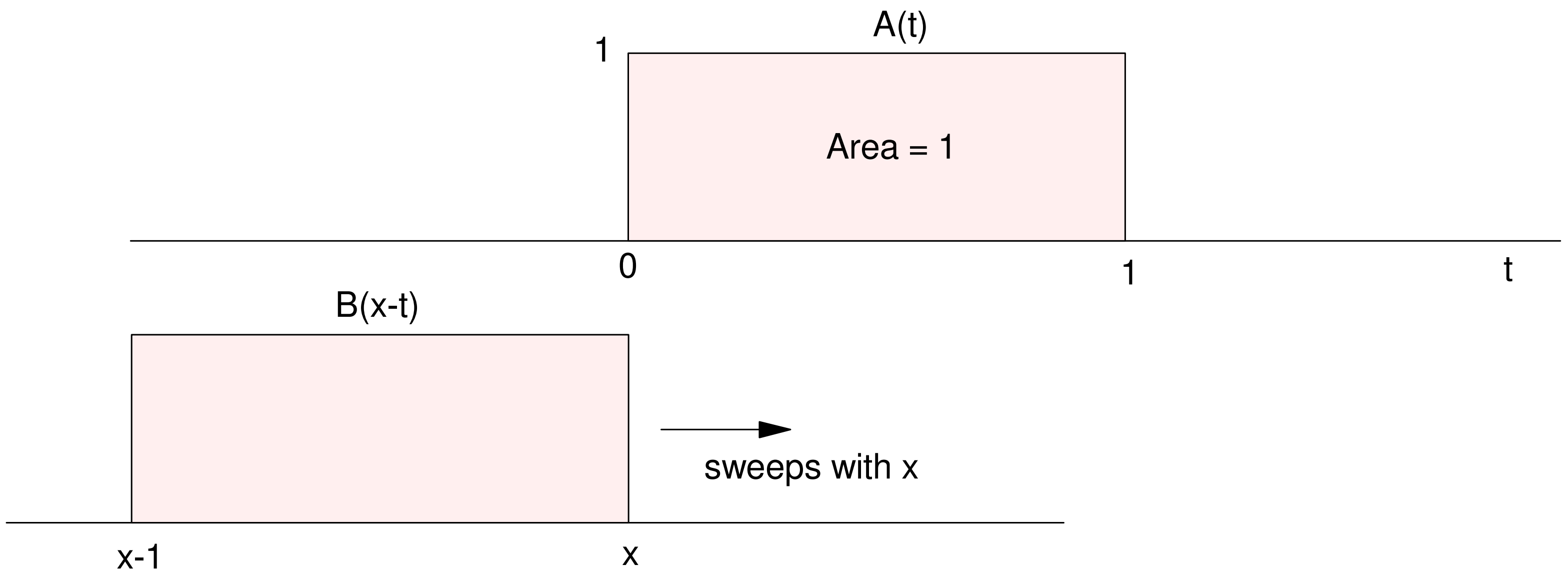
Solution using convolution:

$$A(x) = u(x) - u(x-1)$$

$$B(x) = u(x) - u(x-1)$$

$$Y(x) = A(x) ** B(x)$$

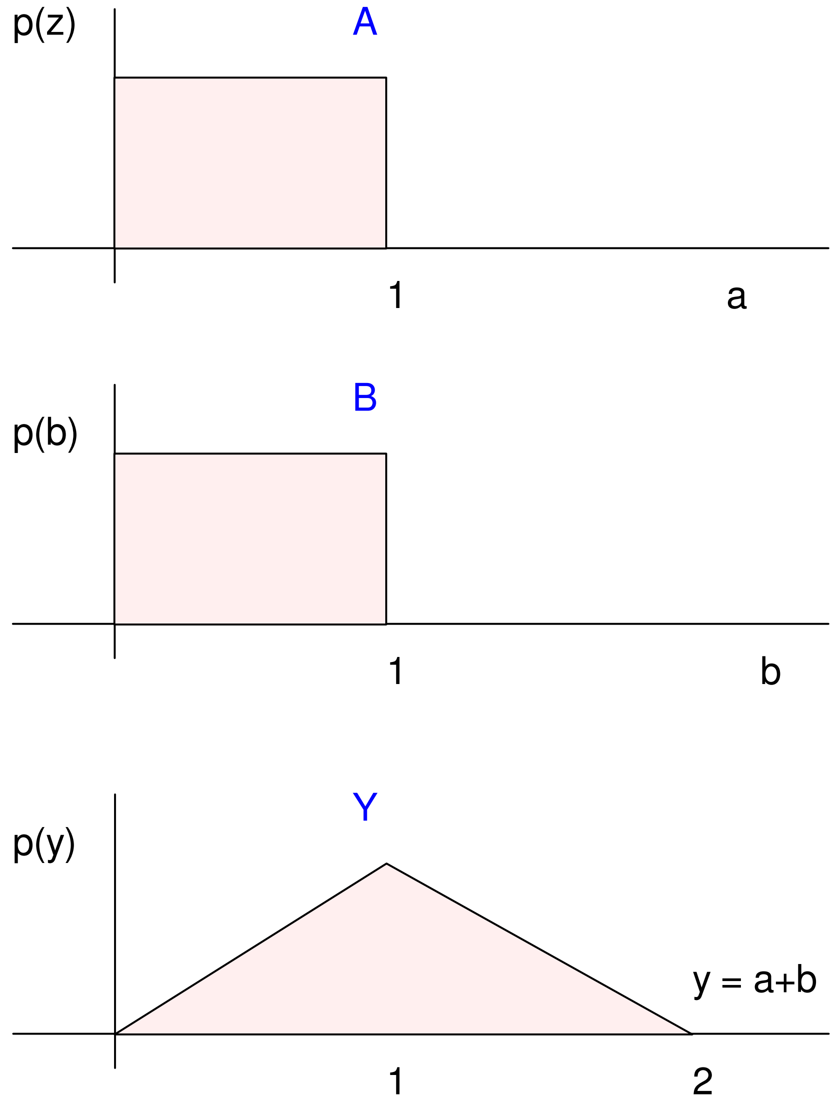
$$Y(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt$$



From the graph, this integral is

Y(x) =

- 0 $\quad x < 0$
- x $\quad 0 < x < 1$
- 1 - x $\quad 1 < x < 2$
- 0 $\quad 2 < x$



Solution using Calculus:

$$Y(x) = \int_{-\infty}^{\infty} A(t)\ B(x-t)\ dt$$

$$Y(x) = \int_{-\infty}^{\infty} (u(t) - u(t-1))\ (u(x-t) - u(x-t-1))\ dt$$

Multiply out giving four terms

$$= \int_{-\infty}^{\infty} u(t)u(x-t)\ dt + \int_{-\infty}^{\infty} -u(t)u(x-t-1))\ dt$$

$$+ \int_{-\infty}^{\infty} -u(t-1)u(x-t))\ dt + \int_{-\infty}^{\infty} u(t-1)\ u(x-t-1))\ dt$$

$$= \int_{0}^{x} 1\ dt - \int_{0}^{x-1} 1\ dt - \int_{1}^{x} 1\ dt + \int_{1}^{x-1} 1 dt$$

$$= xu(x) - (x-1)u(x-1) - (x-1)u(x-1) + (x-2)u(x-2)$$

$$= xu(x) - 2(x-1)u(x-1) + (x-2)u(x-2)$$

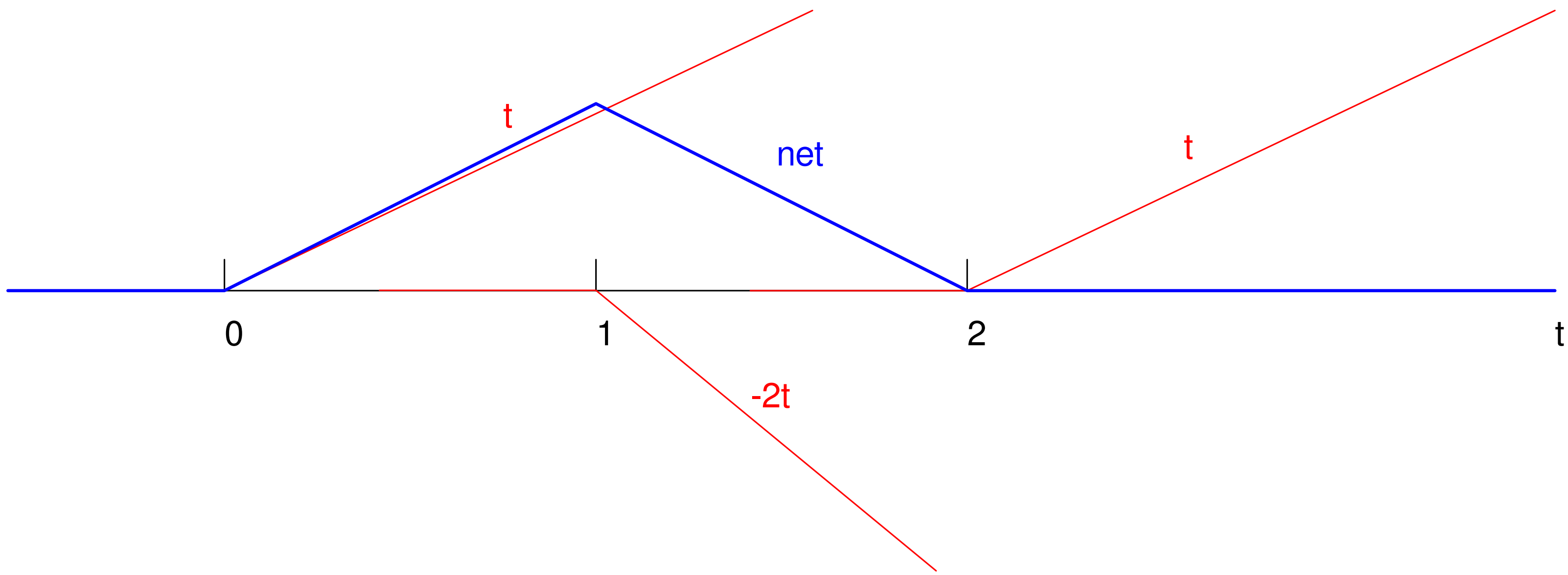
Solve using moment generating functions

$$A(s) = \left(\frac{1}{s}\right)(1 - e^{-s}) \qquad B(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

$$Y(s) = A(s)B(s) \qquad Y(s) = \left(\frac{1}{s^2}\right)(e^{-2s} - 2e^{-s} + 1)$$

Take the inverse LaPlace transform

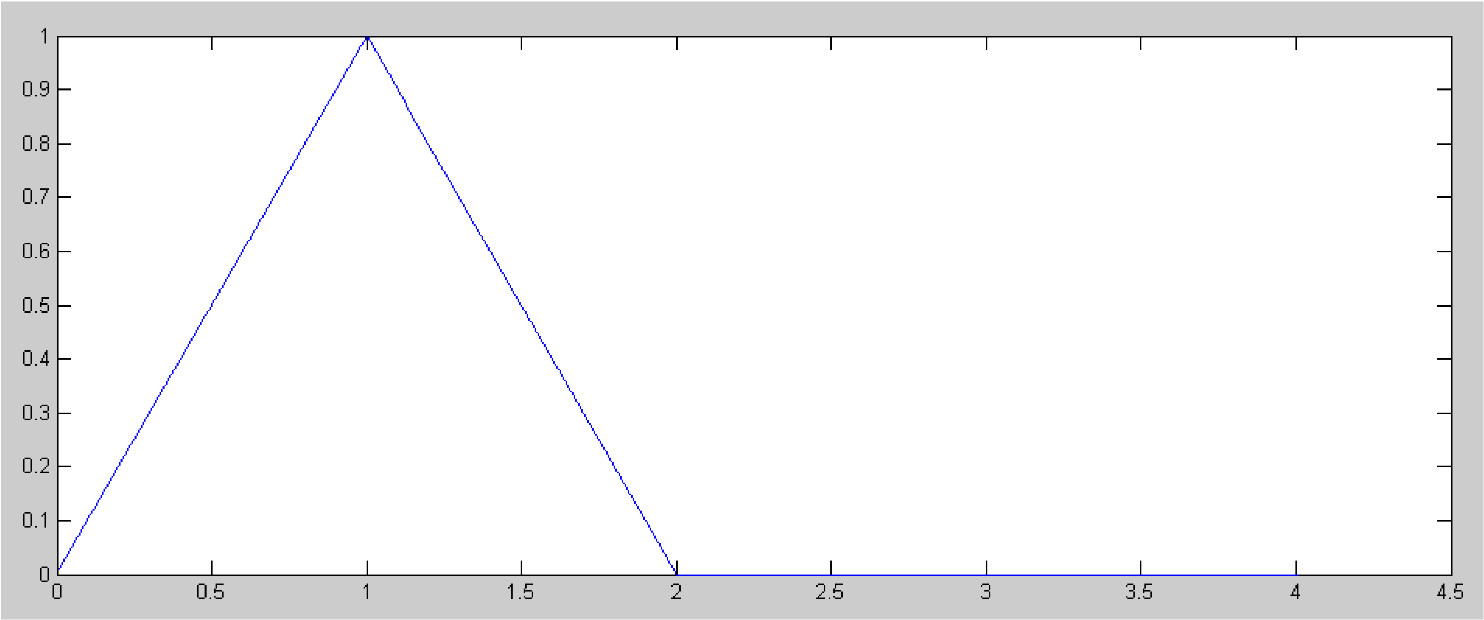
$$y(x) = (x-2)\ u(x-2) - 2(x-1)\ u(x-1) + x\ u(x)$$



Solve in Matlab

- Approximate a uniform distribution with 100 points over the interval (0, 1)

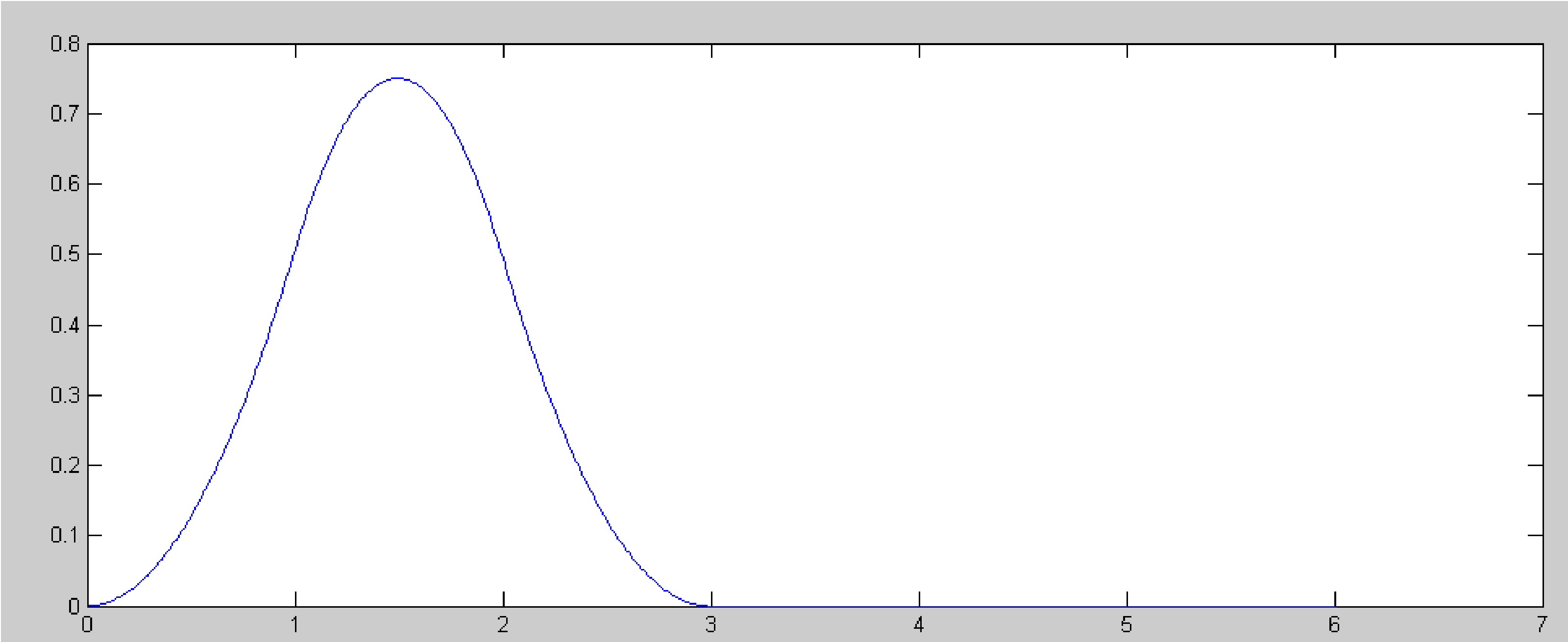
```
dx = 0.01;
x = [0:dx:2]';;
A = 1*(x<1);
B = 1*(x<1);
Y = conv(A, B) * dx;
plot([1:length(Y)]*dx,Y)
```



Example 2: Find the pdf of summing three uniform distribitions

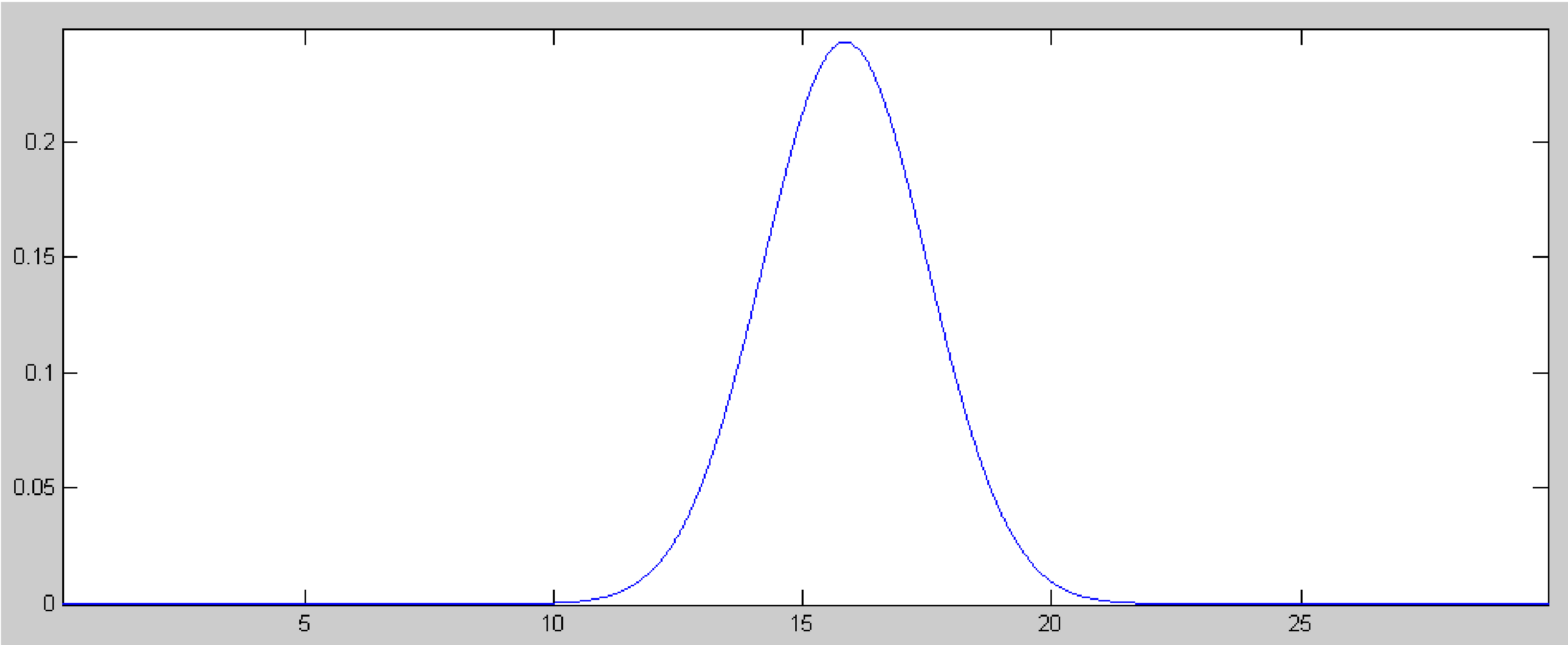
```
x = [0:dx:2]';
A = 1*(x<1);
B = 1*(x<1);
C = 1*(x<1);

Y = conv(A, B) * dx;
Y = conv(Y, C) * dx;
plot([1:length(Y)]*dx,Y)
```



Example 3: Find the pdf for the sum of 32 uniform distributions:

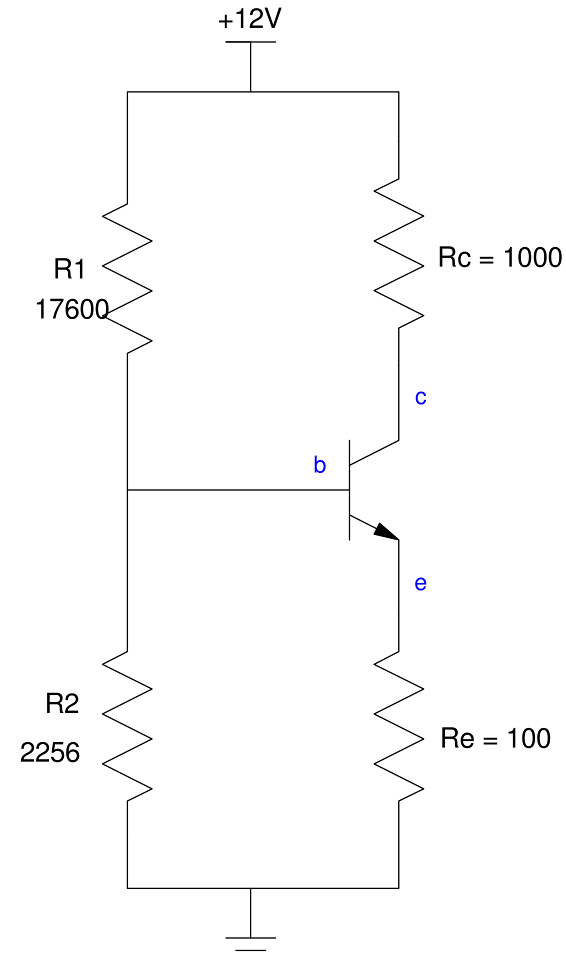
```
x = [0:dx:2]';
A = 1*(x<1);
Y2 = conv(A, A) * dx;
Y4 = conv(Y2, Y2) * dx;
Y8 = conv(Y4, Y4) * dx;
Y16 = conv(Y8, Y8) * dx;
Y32 = conv(Y16, Y16) * dx;
plot([1:length(Y32)]*dx,Y32)
```



Uniform Distribution in Circuit Analysis:

Determine the pdf for Vce. Assume

- Resistors have a 1% tolerance
- The transistor has a gain in the range of 100 to 300



The equations for this circuit (from ECE 321)

$$V_{th} = \left(\frac{R_2}{R_2+R_1}\right)12V$$

$$R_{th} = \left(\frac{R_1 R_2}{R_1+R_2}\right)$$

$$I_b = \left(\frac{V_{th}-.07}{R_{th}+(1+\beta)R_e}\right)$$

$$I_c = \beta I_b$$

$$V_{ce} = 12 - R_c I_c - R_e(I_b + I_c)$$

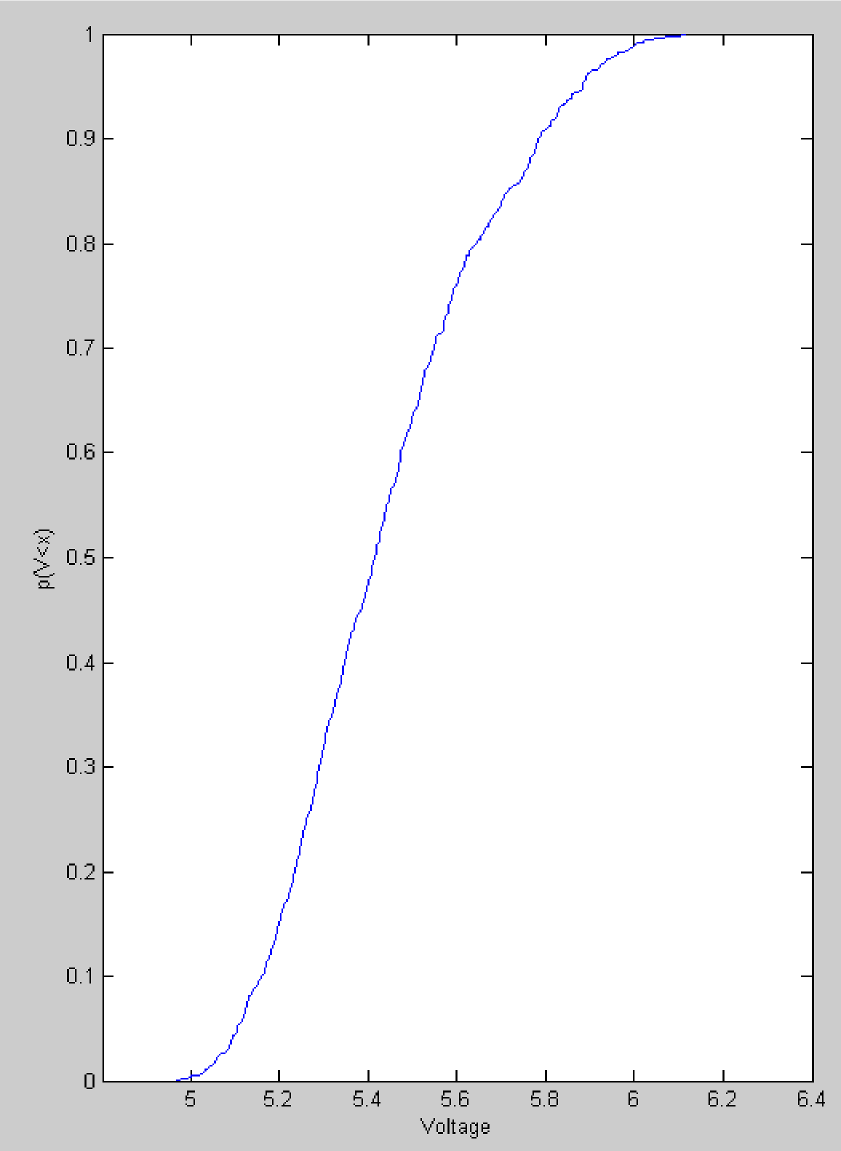
Using a Monte-Carlo simulation

```
DATA = [];

for i=1:1000
 R1 = 17600 * (1 + (rand()*2-1)*0.01);
 R2 = 2256 * (1 + (rand()*2-1)*0.01);
 Rc = 1000 * (1 + (rand()*2-1)*0.01);
 Re = 100 * (1 + (rand()*2-1)*0.01);
 Beta = 200 + 100*(rand()*2-1);
 Vb = 12*(R2 / (R1+R2));
 Rb = 1/(1/R1 + 1/R2);
 Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
 Ic = Beta*Ib;
 Vce = 12 - Rc*Ic - Re*(Ic+Ib);
 DATA = [DATA; Vce];
end

DATA = sort(DATA);

p = [1:length(DATA)]' / length(DATA);
plot(DATA, p)
xlabel('Voltage');
ylabel('p(V<x)')
```

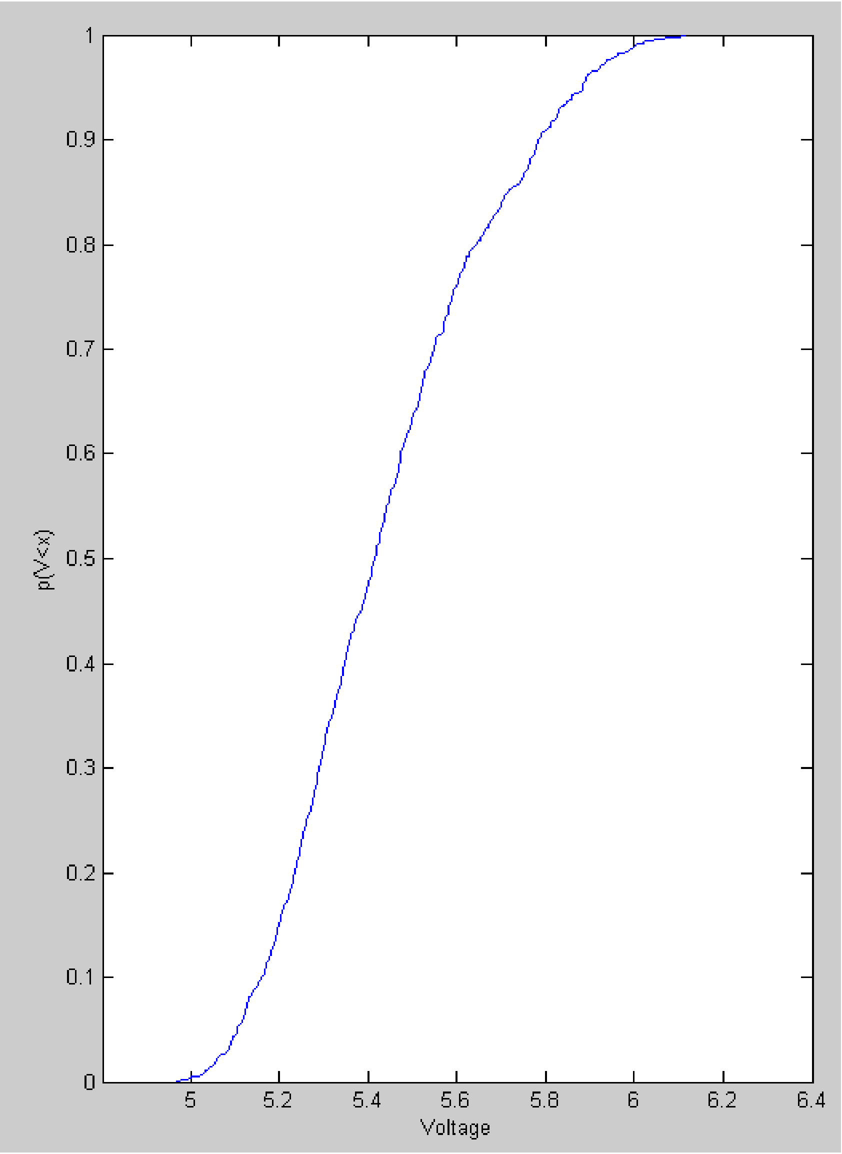


Notes:

- The resulting plot is the cdf of the voltage
- Differentiate it and (in theory) you get the pdf

Due to addition, division, mutipliation, etc. it isn't clear what this distribution is.

- Later, we'll fit a Weibull distribution to this data



Summary

Uniform distrubutions are fairly common and easy to model

- rand() function in Matlab

The sum of uniform distributions converges to a Normal distribution

- Central Limit Theorem (coming soon)

Convolution is required to sum uniform distributions

- conv() in Matlab
- Or you can use multipliation if using LaPlace transforms
  - a.k.a. moment generating functions in statistics