Continuous Probability Density Functions

ECE 341: Random Processes

Lecture #11

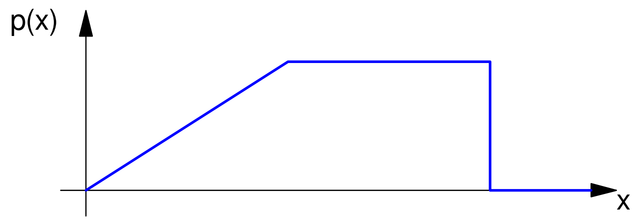
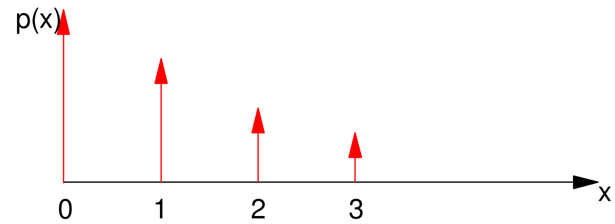
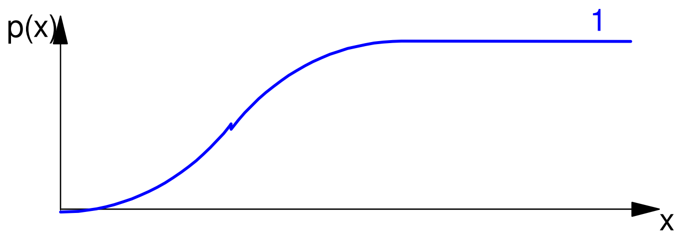
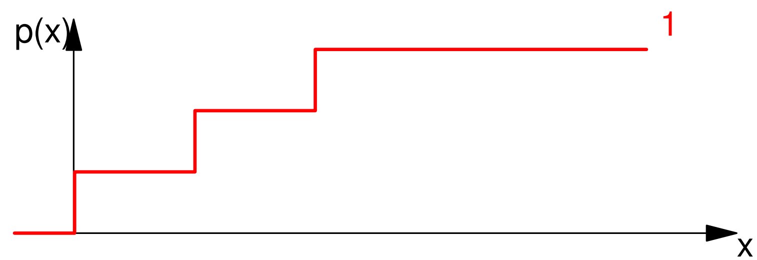
note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Continuous Probability Density Functions

- The value of a 1k resistor
- The gain of a 3904 transistor
- The resistance of a thermistor at 25C
- The hottest temperature in May,
- etc.

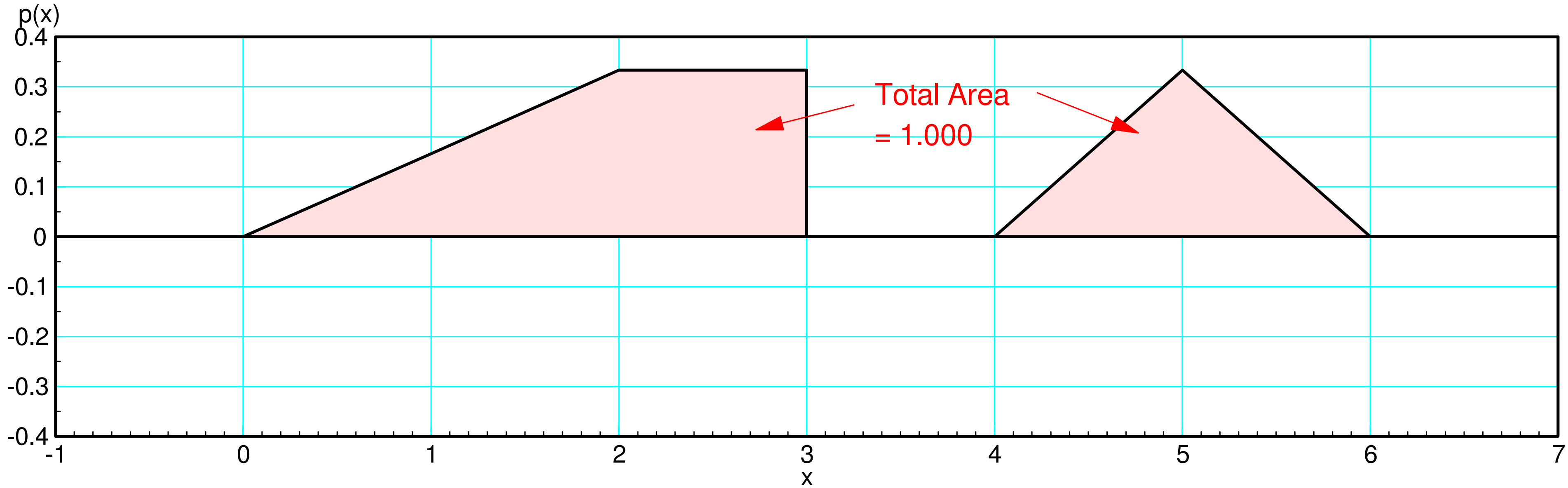
For each of these cases, the variable can assume any value over a range.

Most of what we did with discrete-time probabilities apply to continuous time - with just a slight change.

	Continuous	**Discrete**
pdf probability density function pdf = derivative of cdf		
cdf cumulative density function cdf = integral of pdf		
mgf moment generating function	$mgf = \psi(s)$	$mgf = \psi(z)$
m0 Area = 1	$m_0 = \psi(s=0) = 1$	$m_0 = \psi(z=1) = 1$
m1 mean	$m_1 = \psi'(s=0)$	$m_1 = -\psi'(z=1)$
m2	$m_2 = \psi''(s=0)$	$m_2 = \psi''(z=1)$
variance	$\sigma^2 = m_2 - m_1^2$	$\sigma^2 = m_2 - m_1^2$

Example of a Continuous pdf

- Total area must be 1.000



Example pdf: p(x)

Implementation in Matlab

To implement this pdf, it's actually easier to use the cdf (the integral of the pdf)

$$c(x) = \int_{-\infty}^{x} p(y)\, dy$$

Expressing p(x) mathematically:

$$p(x) = \begin{cases} 0 & x < 0 \\ x/6 & 0 < x < 2 \\ 1/3 & 2 < x < 3 \\ 0 & 3 < x < 4 \\ (x-4)/3 & 4 < x < 5 \\ (6-x)/3 & 5 < x < 6 \\ 0 & 6 < x \end{cases}$$

The cdf is then the integral of the pdf

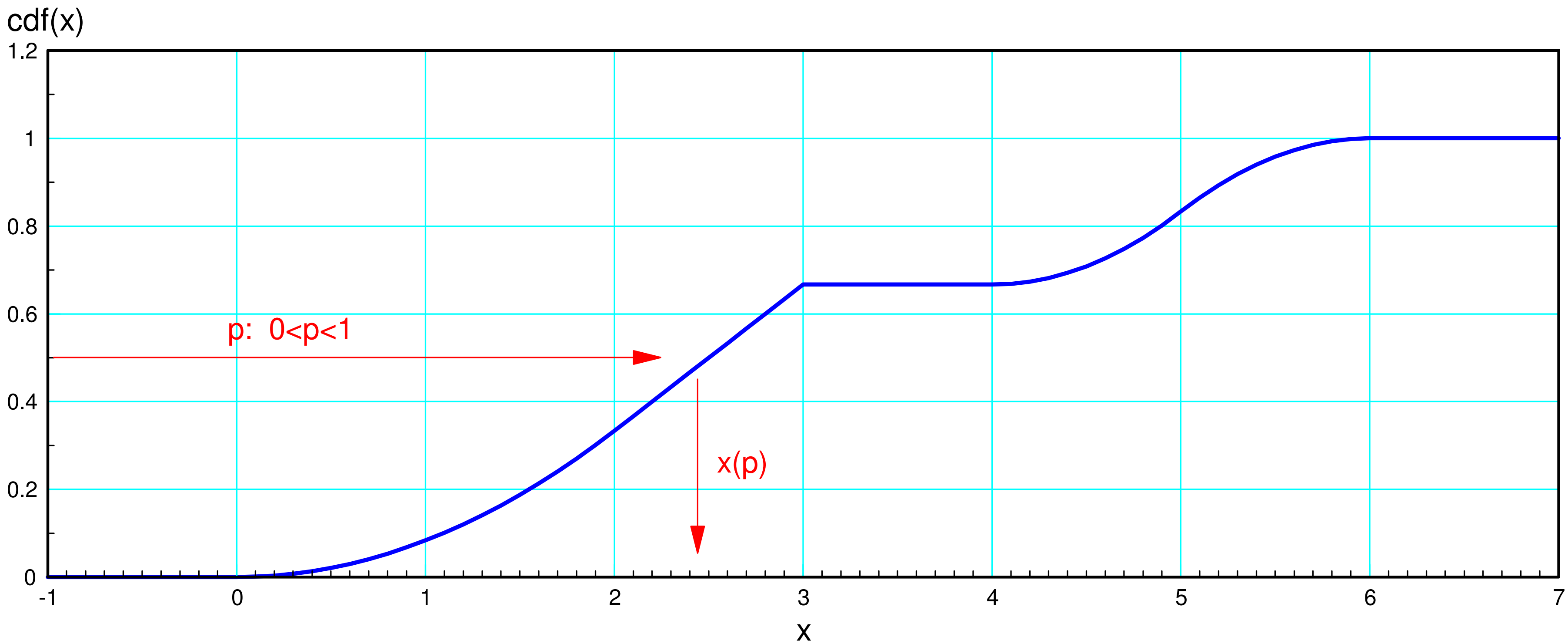
- The integration constant maintains continuity
- Note that the cdf must wind up at 1.000 (the probability of x being something is 100%)

$$cdf(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{12} & 0 < x < 2 \\ \frac{x-2}{3} + \frac{1}{3} & 2 < x < 3 \\ \frac{2}{3} & 3 < x < 4 \\ \frac{(x-4)^2}{6} + \frac{2}{3} & 4 < x < 5 \\ 1 - \frac{(6-x)^2}{6} & 5 < x < 6 \\ 1 & 6 < x \end{cases}$$

In Matlab you can express this using if statements

```
function [ y ] = cdf( x )
if ( x < 0)
    y = 0;
elseif (x < 2)
    y = x*x/12;
elseif (x < 3)
    y = (x-2)/3 + 1/3;
elseif (x < 4)
     y = 2/3;
elseif (x < 5)
     y = (x-4)^2 /6 + 2/3;
elseif (x < 6)
    y = 1 - (6 - x)^2 / 6;
else y = 1;
end
```

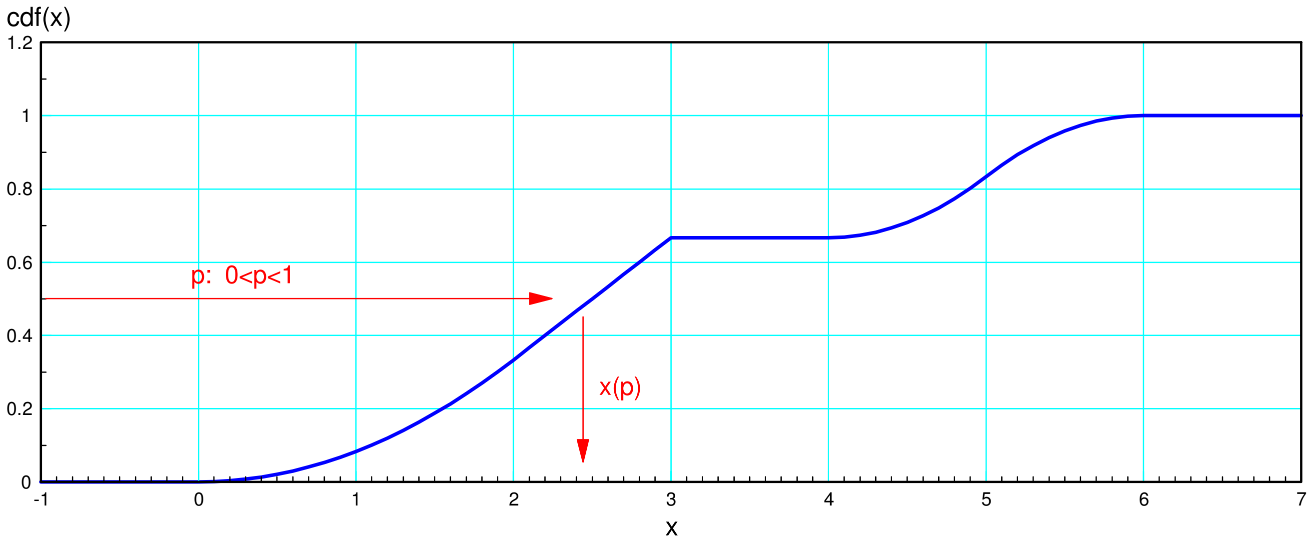
```
x = [-1:0.1:10]';
y = 0*x;

for i=1:length(x)
   y(i) = cdf(x(i));
   end

plot(x,y)
```



To generate x,

- Determine the probability, p. Note that $0 < p < 1$
- The Y coordinate is p. Use the cdf to map p to x(p)

$$x(p) = \begin{cases} \sqrt{12p} & 0 < p < 1/3 \\ 3p + 1 & 1/3 < p < 2/3 \\ 4 + \sqrt{6p - 4} & 2/3 < p < 5/6 \\ 6 - \sqrt{6(1-p)} & 5/6 < p < 1 \end{cases}$$

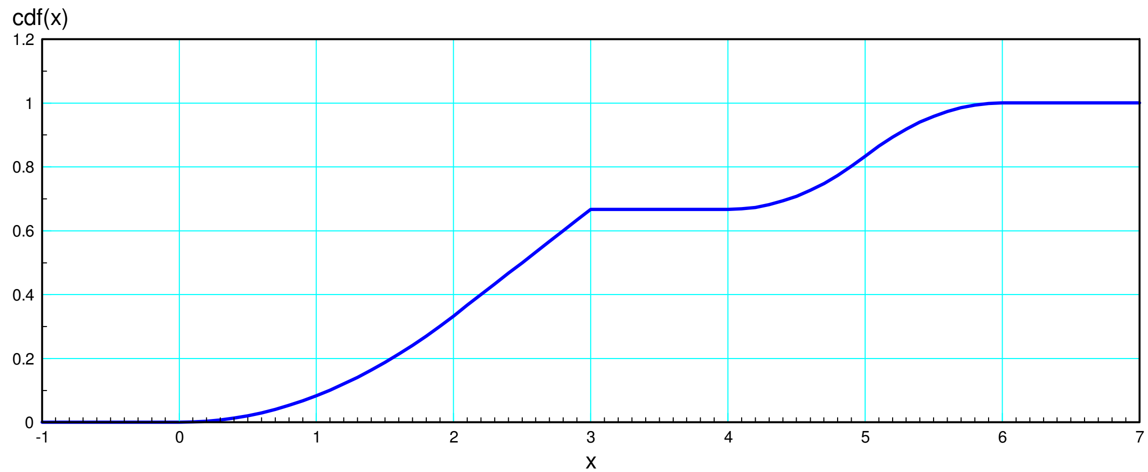


In matlab:

```
function [ x ] = pdf( p )
if ( p < 1/3)
    x = sqrt(12*p);
elseif (p < 2/3)
    x = 3*p+1;
elseif (p < 5/6)
    x = 4 + sqrt(6*p-4);
else
    x = 6 - sqrt(6*(1-p));
end
```

As a check, see if you get the graph as before:

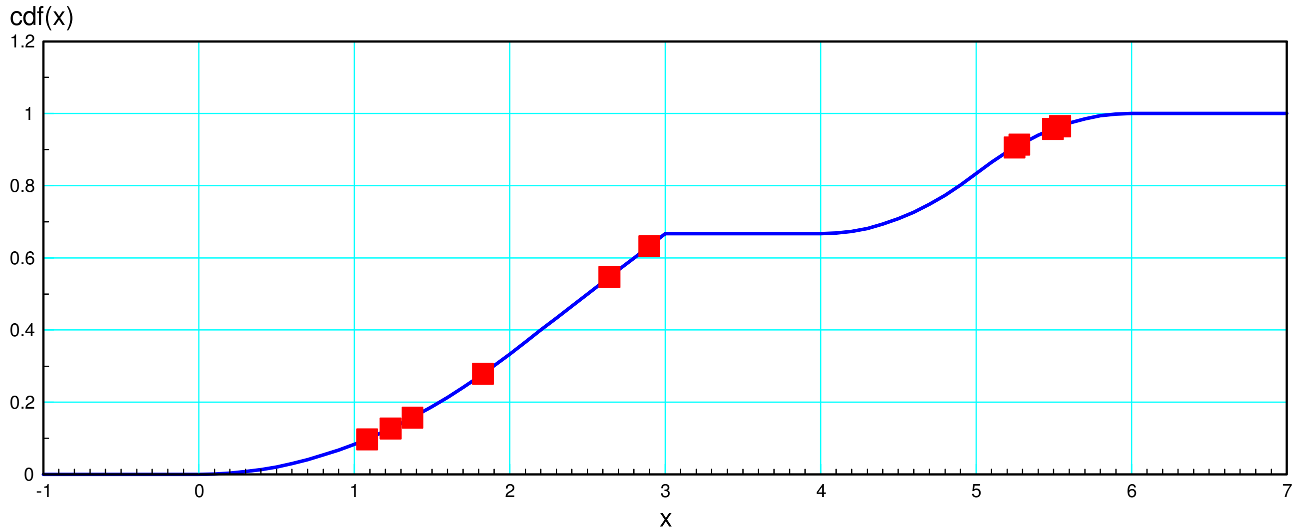
```
p = [0:0.01:1]';
x = 0*p;
for i=1:length(p)
   x(i) = pdf(p(i));
   end
plot(x,p)
```



So now with this function, we can generate random values of x:

```
for i=1:10
   p = rand;
   x = pdf(p);
   disp([p,x])
   end
```

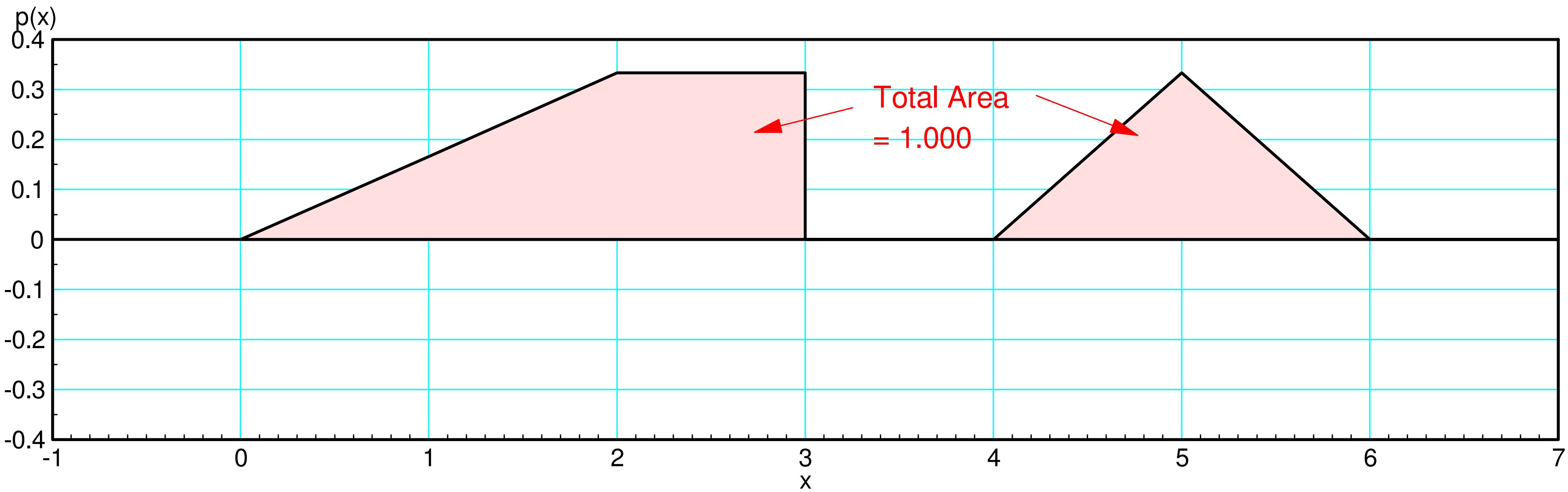
p	x
0.9058	5.2482
0.1270	1.2344
0.9134	5.2791
0.6324	2.8971
0.0975	1.0819
0.2785	1.8281
0.5469	2.6406
0.9575	5.4951
0.9649	5.5410
0.1576	1.3753



Moment Generating Function

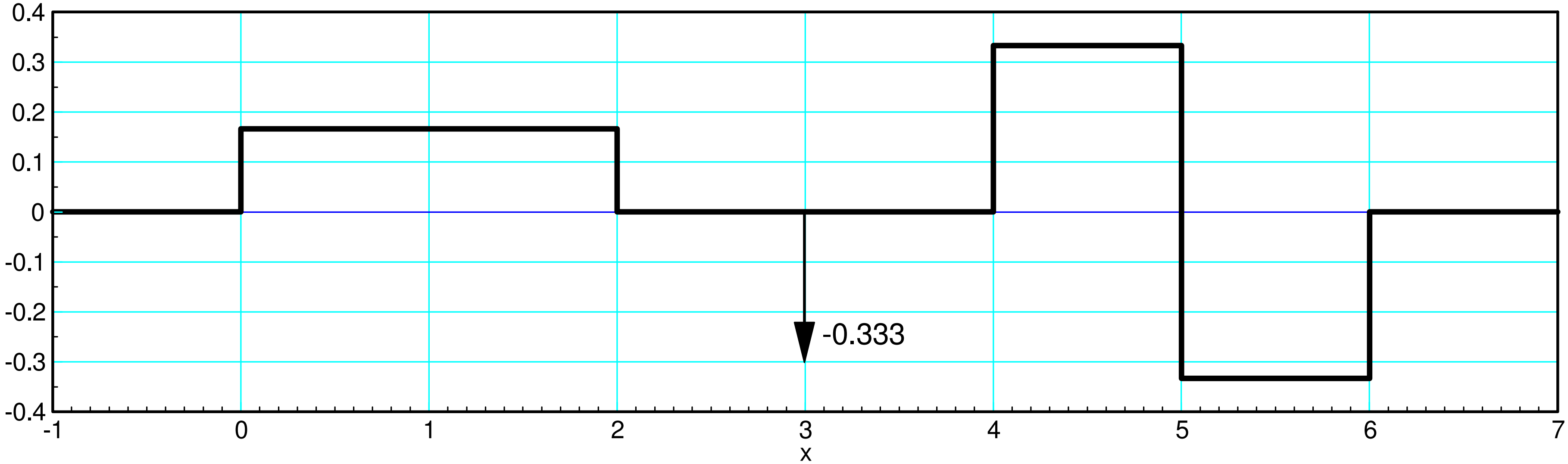
- Find the LaPlace transform of the pdf
- The result is the moment generating function

Example:



Take the derivative

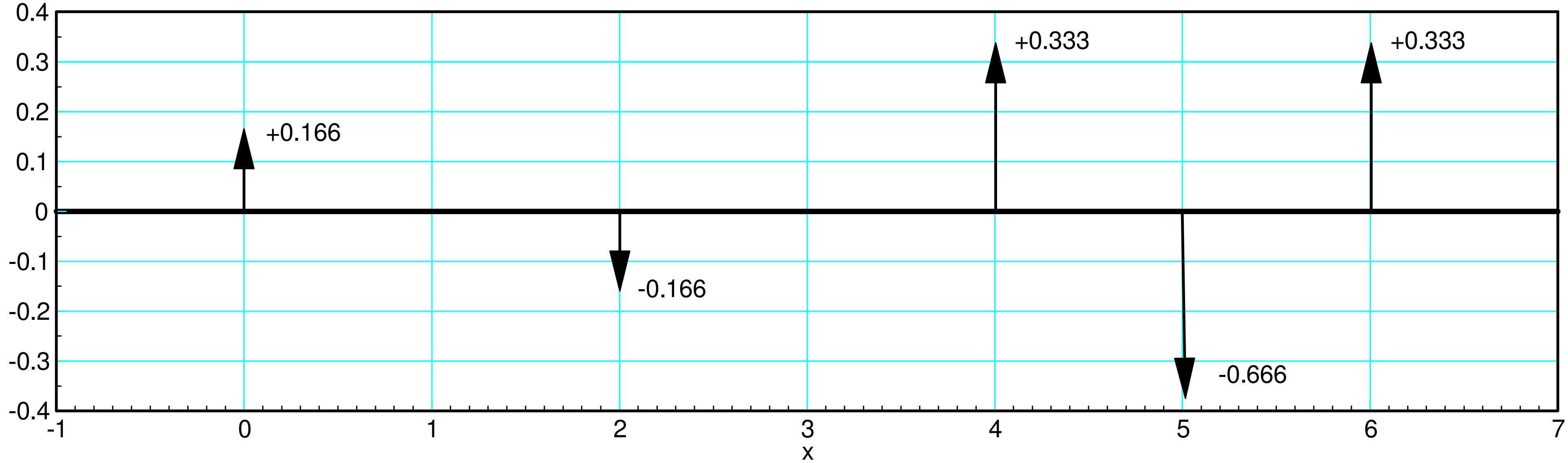
- goal: end up with something we recognize (a delta function typically)
- differentiation is multiplying by 's'
- $L(\text{delta}) = -0.333\, e^{-3s}$



first derivative of pdf.   LaPlace tranform of delta function at x=3 is easy...

Take another derivative

- $L(p'') = 0.166 - 0.166e^{-2s} + 0.333e^{-4s} - 0.666e^{-5s} + 0.333e^{-6s}$



Second derivative of pdf

Integrate to get back to the pdf

- Integation is dividing by 's'

$$\psi(s) = L(p) + \frac{L(p')}{s} + \frac{L(p'')}{s^2}$$

$$\psi(s) = \left(\frac{-0.333e^{-3s}}{s}\right) + \left(\frac{0.166-0.166e^{-2s}+0.333e^{-4s}-0.666e^{-5s}+0.333e^{-6s}}{s^2}\right)$$

This is the moment generating funciton for the previous pdf

Summary

Moment Generating Functions are really just the LaPlace transform of the pdf

They are useful for

- Generating moments (duh), and
- Avoiding convolution of pdf's

The latter will show up in the following lectures.