LaPlace Transforms

ECE 341: Random Processes

Lecture #10

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & otherwise \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer funcitons and back.

Example 1: Find the transfer function from X to Y

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3Y + 6s^2Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)X$$

The transfer function from X to Y is

$$\boxed{G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)}$$

- Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.

Example 2: Find the differential equation relating X and Y

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right) X$$

Cross multiply:

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

Note that 'sY' means 'the derivative of Y'

$$\boxed{\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x}$$

Sidelight: Fractional powers are not allowed in transfer functions.

- $s^2 Y$ means 'the second derivative of Y'.
- $s^{2.3} Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find y(t) given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right) X$$

and

$$x(t) = 3\cos(4t)$$

Solution: From Euler's identity:

$$e^{j4t} = \cos(4t) + j\sin(4t)$$

$$(a+jb)\, e^{j4t} = (a\cos(4t) - b\sin(4t)) + j(\cdot)$$

$$a + jb \leftrightarrow a\cos(4t) - b\sin(4t)$$

- real = cosine
- -imag = sine

Take the real part and you the sine wave.

$$s = j4$$

$$X = 3 + j0$$

Convert to phasors

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right) X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j4} (3 + j0)$$

$$Y = (-0.191 - j0.315)(3 + j0)$$

$$Y = -0.544 - j0.946$$

meaning

$$y(t) = -0.544 \cos(4t) + 0.946 \sin(4t)$$

Example 4: Find y(t) if

$$x(t) = 5\cos(20t) + 6\sin(20t)$$

Solution:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6}\right)_{s=j20}(5-j6)$$

$$Y = (-0.019 - j0.005)(5-j6)$$

$$Y = -0.123 + j0.092$$

meaning

$$y(t) = -0.123\cos(20t) - 0.092\sin(20t)$$

Note that the gain varies with frequency (i.e. this is a filter).

Example 5: Find y(t) if

$$x(t) = 3\cos(4t) + 5\cos(20t) + 6\sin(20t)$$

Solution: Treat this as two separate problems

- $x(t) = 3\cos(4t)$
- $x(t) = 5\cos(20t) + 6\sin(20t)$

The total input is the sum of the two x(t)'s.

The total output is the sum of the two y(t)'s

$$\begin{aligned} y(t) = &-0.544\cos(4t) + 0.946\sin(4t) \\ &-0.123\cos(20t) - 0.092\sin(20t) \end{aligned}$$

Solving Transfer Functions with Step Inputs

Common LaPlace Transforms		
Name	Time: y(t) (t >0)	LaPlace: Y(s)
delta (impulse)	$\delta(t)$	1
unit step	1	$\left(\frac{1}{s}\right)$
unit ramp	t	$\left(\frac{1}{s^2}\right)$
unit parabola	t^2	$\left(\frac{2}{s^3}\right)$
decaying exponential	e^{-bt}	$\left(\frac{1}{s+b}\right)$
	$t\ e^{-bt}$	$\left(\frac{1}{(s+b)^2}\right)$
	$t^2\ e^{-bt}$	$\left(\frac{2}{(s+b)^3}\right)$
	$2a \cdot e^{-bt}\cos(ct-\theta)u($	$\left(\frac{a\angle\theta}{s+b+jc}\right)+\left(\frac{a\angle-\theta}{s+b-jc}\right)$

Example: Find the impulse response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right) X$$

$$Y = \left(\frac{5}{s+3}\right) (1)$$

From the above table:

$$y(t) = 5e^{-3t}u(t)$$

Example: Find the step response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)\left(\frac{1}{s}\right) = \left(\frac{5}{s(s+3)}\right)$$

Do a partial fraction expansion

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{5/3}{s}\right) - \left(\frac{5/3}{s+3}\right)$$

Using the above table for each term:

$$\boxed{y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t}\right)u(t)}$$

Repeated Roots

Find the step response of

$$Y = \left(\frac{1}{(s+1)^2}\right) X$$

Solution: Use the table or change the problem

$$Y = \left(\frac{1}{(s+1.01)(s+0.99)}\right) X$$

There are no longer repeated roots

The differential equation is almost identical

- It's just a model for a real system
- No model is perfect
- The odds of two poles being *exactly* the same is almost zero

Solving with Complex Roots:

$$\left(\frac{a\angle\theta}{s+b+jc}\right)+\left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a\cdot e^{-bt}\cos(ct-\theta)u(t)$$

Example: Find the y(t) given that

$$Y(s) = G\cdot U = \left(\frac{15}{s^2+2s+10}\right)\cdot\left(\frac{1}{s}\right)$$

Solution: Factoring Y(s)

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right)+\left(\frac{0.7906\angle-161.56^0}{s+1+j3}\right)+\left(\frac{0.7906\angle 161.56^0}{s+1-j3}\right)$$

$$y(t) = 1.5+1.5812\cdot e^{-t}\cdot\cos\left(3t+161.56^0\right) \qquad \text{for t>0}$$

Properties of LaPlace Transforms

Linearity: $af(t) + bg(t) \Leftrightarrow aF(s) + bG(s)$

Convolution: $f(t) ** g(t) \Leftrightarrow F(s) \cdot G(s)$

Differentiation:

$$\frac{dy}{dt} \Leftrightarrow sY - y(0)$$

$$\frac{d^2y}{dt^2} \Leftrightarrow s^2 Y - sy(0) - \frac{dy(0)}{dt}$$

Integration: $\int_0^t x(\tau)d\tau = \frac{1}{s} X(s)$

Delay $x(t-T) \Leftrightarrow e^{-sT} X(s)$

Proofs

Linearity:

$$L(af(t)+bg(t)) = \int_{-\infty}^{\infty} (af(t)+bg(t)) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} (af(t)) \cdot e^{-st} \cdot dt + \int_{-\infty}^{\infty} (bg(t)) \cdot e^{-st} \cdot dt$$

$$= a \int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt + b \int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt$$

$$= aF(s) + bG(s)$$

Convolution:

$$f(t) * * g(t) = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau$$

$$L(f(t) * * g(t)) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot d\tau \right) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot e^{-st} \cdot dt \right) \cdot d\tau$$

$$= \left(\int_{-\infty}^{\infty} f(t-\tau) \cdot e^{-st} \cdot dt \right) \cdot \left(\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \right)$$

$$= F(s) \cdot G(s)$$

Differentiation:

$$L\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Assume causal (zero for t<0)

$$L\left(\frac{dx}{dt}\right) = \int_{0}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\int a' \cdot b \cdot dt = ab - \int a \cdot b' \cdot dt$$

Let

$$a' = \frac{dx}{dt}$$

$$a = x$$

$$b = e^{-st}$$

then

$$
\begin{aligned}
L\left(\frac{dx}{dt}\right) &= \int_0^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt \\
&= (x \cdot e^{-st})_0^{\infty} - \int_{-\infty}^{\infty} -s \cdot x(t) \cdot e^{-st} \cdot dt \\
&= -x(0) + s \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt \\
&= sX - x(0)
\end{aligned}
$$

Integration:

$$L\left(\int_0^t x(\tau)\cdot d\tau\right) = \int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

Integrate by parts.

$$\int a\cdot b'\cdot dt = ab - \int a'\cdot b\cdot dt$$

Let

$$a = \int_0^t x(\tau)\cdot d\tau$$

$$b' = e^{-st}$$

then

$$a' = x$$

$$b = \tfrac{-1}{s}e^{-st}$$

$$L\left(\int_0^t x(\tau)\cdot d\tau\right) = \int_{-\infty}^{\infty}\left(\int_0^t x(\tau)\cdot d\tau\right)\cdot e^{-st}\cdot dt$$

$$= \left(\int_0^t x(\tau) \cdot d\tau \cdot \frac{-1}{s} e^{-st}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \cdot \frac{-1}{s} e^{-st} \cdot dt$$

Assuming the function vanishes at infinity

$$= \frac{1}{s} \int_{-\infty}^{\infty} x \cdot dt$$

$$= \left(\frac{1}{s}\right) X(s)$$

Time Delay

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(t-T) \cdot e^{-st} \cdot dt$$

Do a change of variable

$$t - T = \tau$$

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s(\tau+T)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot e^{-sT} \cdot d\tau$$

$$= e^{-sT} \cdot \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot d\tau$$

$$= e^{-sT} \cdot X(s)$$

Summary

LaPlace transforms

- Turn differential equations into algebraic equations
- Turn convolution in to multiplication

They are very useful in solving differential equations

- i.e. Analog Signals
- Multiplications is easier than convolution

They will also be useful in Random Processes

- If we ever need to convolve continuous pdf's, LaPlace transforms will turn this into multiplication
- In statistics, LaPlace transforms are termed *moment generating functions*