Pascal Distribution

ECE 341: Random Processes

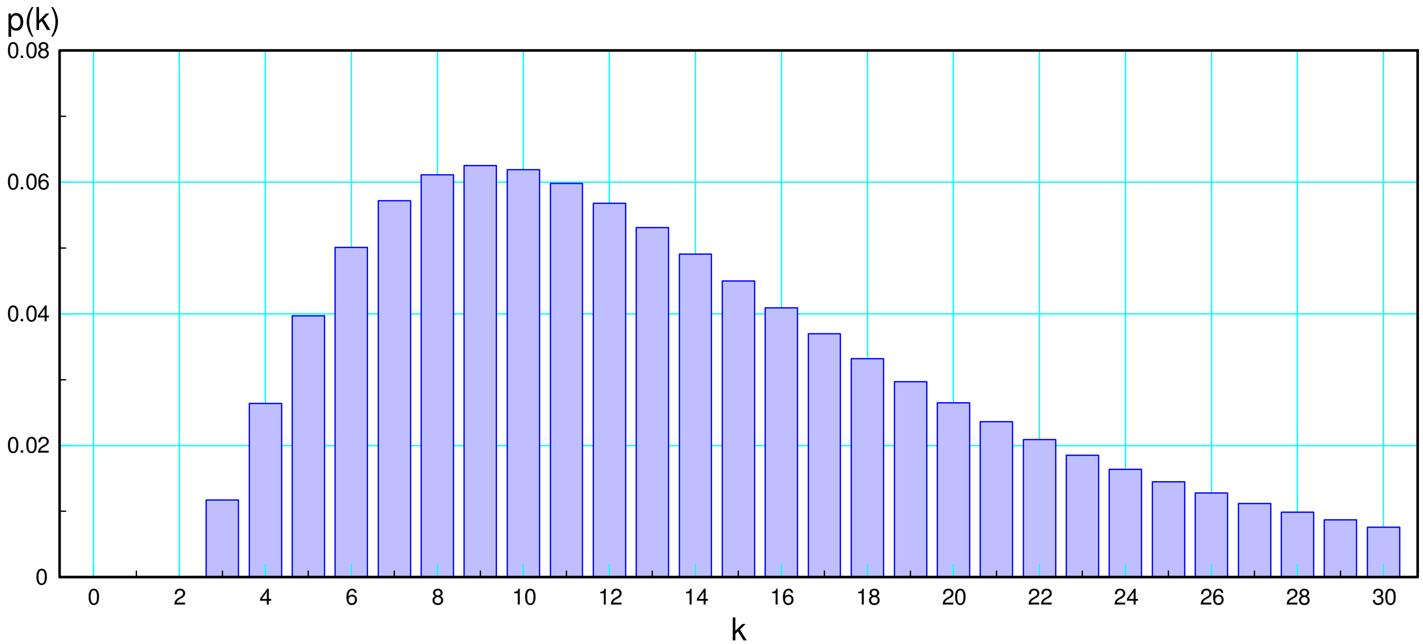
Lecture #9

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Pascal Distribution (a.k.a. Negative Binomial Distribution)

A Pascal distribution is similar to a geometric distribution, modeling

- The number of times you roll a die until you get r ones
- The number of times you make a trip with a car until r things fail (and it's time to buy a new car)
- The number of days until r accidents happen at work and you're promoted (Peter principle)



pdf - mgf - mean - variance

Distribution	description	pdf	mgf	mean	variance
Bernoulli trial	flip a coin obtain m heads	$p^m q^{1-m}$	$q+p/z$	p	p(1-p)
Binomial	flip n coins obtain m heads	$\binom{n}{m} p^m q^{n-m}$	$(q+p/z)^n$	np	np(1-p)
Hyper Geometric	Bernoulli trial without replacement	$\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$			
Uniform range = (a,b)	toss an n-sided die	$1/n \quad a \le m \le b$ $0 \quad otherwise$	$\left(\frac{1+z+z^2+\ldots+z^{n-1}}{n\, z^b}\right)$	$\left(\frac{a+b}{2}\right)$	$\left(\frac{(b+1-a)^2-1}{12}\right)$
Geometric	Bernoulli until 1st success	$p\, q^{k-1}$	$\left(\frac{p}{z-q}\right)$	$\left(\frac{1}{p}\right)$	$\left(\frac{q}{p^2}\right)$
Pascal	Bernoulli until rth success	$\binom{x-1}{r-1} p^r q^{x-r}$	$\left(\frac{p}{z-q}\right)^r$	$\left(\frac{r}{p}\right)$	$\left(\frac{qr}{p^2}\right)$

Source: Wikipedia

pdf for a Pascal Distribution

Assume you toss a coin with the probability of heads being p.

The probability of getting r heads on kth flip is:

- On the kth flip, you must get a heads (probability = p):
- On the previous k-1 flips, you got r-1 heads. (binomial distribution)

$$f(k) = \binom{k-1}{r-1} p^{r-1} q^{(k-1)-(r-1)}$$

- Both must happen

$$f(k) = p \cdot \binom{k-1}{r-1} p^{r-1} q^{(k-1)-(r-1)} = \binom{k-1}{r-1} p^{r} q^{k-r}$$

Mean and Variance

The moment generating function for a geometric distibution is

$$\psi(z) = \left(\frac{p}{z-q}\right)$$

The moment generating for r geometric distributions (a Pascal distribution) is

$$\psi(z) = \left(\frac{p}{z-q}\right)^r$$

Zeroth Moment

$$m_0 = \psi(z-1) = 1$$

$$m_0 = \left(\frac{p}{1-q}\right)^r = 1^r = 1$$

This is a valid moment generating function (probabities add to one)

1st Moment (mean)

$$m_1 = -\psi'(z = 1)$$

$$m_1 = -\frac{d}{dz}\left(\left(\frac{p}{z-q}\right)^r\right)_{z=1}$$

$$m_1 = -\left(\frac{-r \cdot p^r}{(z-q)^{r+1}}\right)_{z=1}$$

$$m_1 = \left(\left(\frac{r \cdot p^r}{(p)^{r+1}}\right)\right) = \left(\frac{r}{p}\right)$$

$$\mu = m_1 = \left(\frac{r}{p}\right)$$

Second Moment

$$m_2 = \psi''(z = 1)$$

$$m_1 = \frac{d^2}{dz^2}\left(\left(\frac{p}{z-q}\right)^r\right)_{z=1}$$

$$m_2 = \left(\frac{d}{dz}\left(\frac{-r \cdot p^r}{(z-q)^{r+1}}\right)\right)_{z=1}$$

$$m_2 = \left(\left(\frac{r(r+1)p^r}{(z-q)^{r+2}}\right)\right)_{z=1}$$

$$m_2 = \left(\left(\frac{r(r+1)p^r}{p^{r+2}}\right)\right)$$

$$m_2 = \left(\frac{r(r+1)}{p^2}\right)$$

Variance:

$$\sigma^2 = m_2 - m_1^2$$

$$\sigma^2 = \left(\frac{r(r+1)}{p^2}\right) - \left(\frac{r}{p}\right)^2$$

$$\sigma^2 = \left(\frac{r}{p^2}\right)$$

and again I'm off by a factor of q...

$$\sigma^2 = \left(\frac{qr}{p^2}\right)$$

Example 1:

- Rolling a die until you get a 1 or 2 (p = 1/3)
- The number of times you do the dishes until someone notices (p = 1/3)
- The number of parties you go to until you catch COVID-19 (assume p = 1/3)

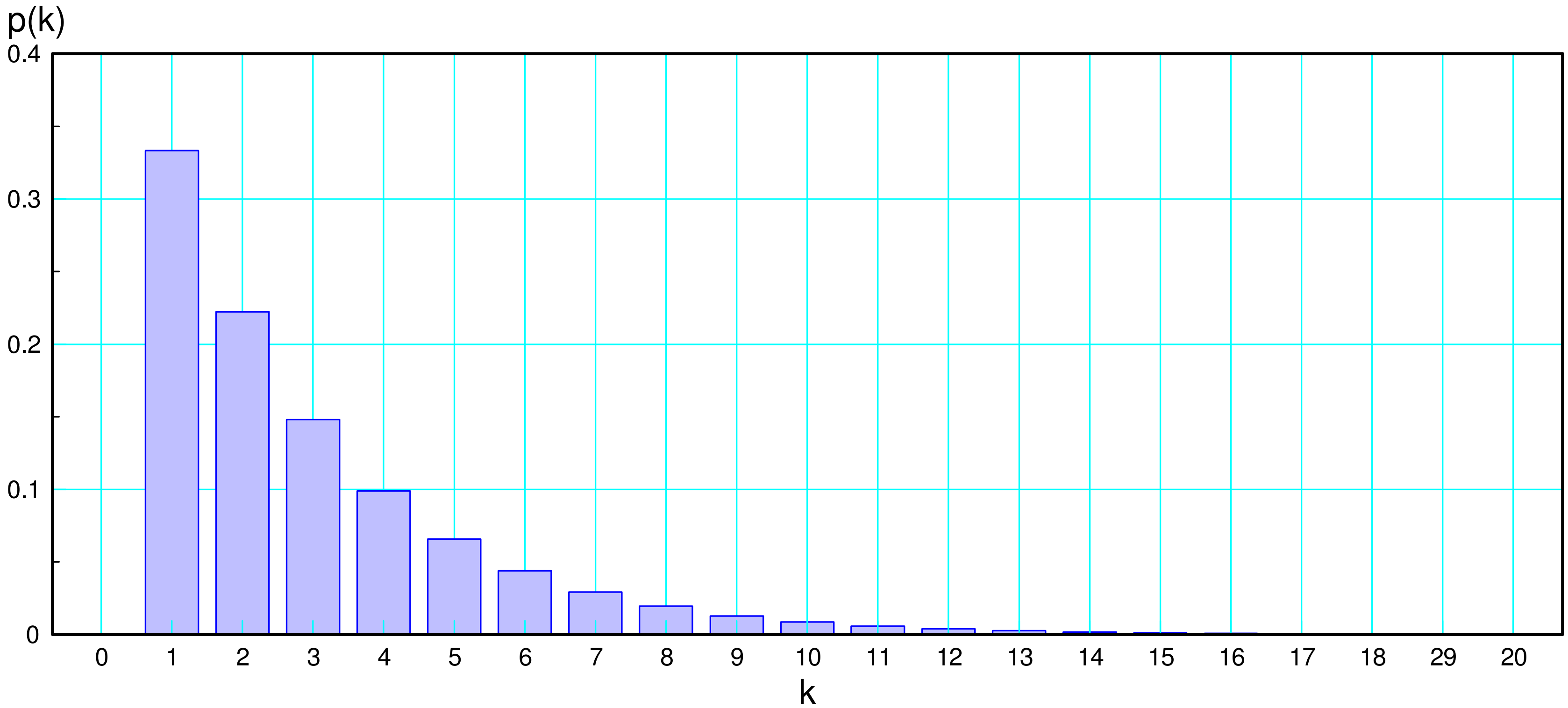
Solution: This is a geometric distribution

$$p(k) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} u(k-1)$$

Matlab Code:

```
k = [1:30]';
A = (1/3) * (2/3).^(k-1) .* (k-1 >= 0);
k = [0;k];
A = [0;A];

bar(k,A)
```

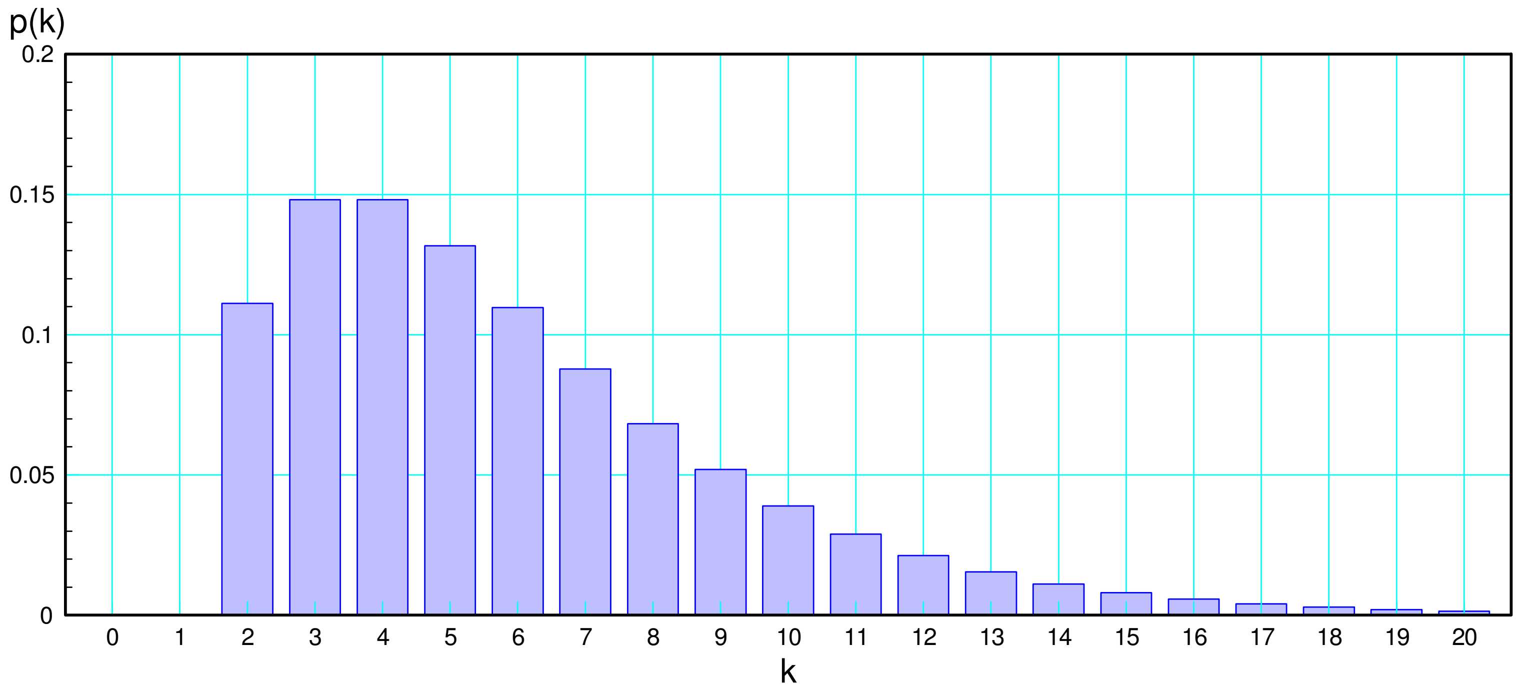


pdf for the number of rolls (k) until you roll a 1 or 2 (geometric distribution)

Example 2: Determine the pdf for

- The number of times you roll a 6-sided die until you roll a 1 or 2 twice (p = 1/3)
- The number of times you do the dishes until two people notice (p = 1/3)
- The number of parties you go to until you have two exposes to COVID-19 (assume p = 1/3)

Solution: This is a Pascal distribution.



Option 1: Convolution in matlab. We have the pdf for r = 1 from before. Use convolution to repeat the event

```
A2 = conv(A,A);
[k(1:21), A2(1:21)]

       k          A2
          0          0
     1.0000          0
     2.0000     0.1111
     3.0000     0.1481
     4.0000     0.1481
     5.0000     0.1317
     6.0000     0.1097
     7.0000     0.0878
     8.0000     0.0683
     9.0000     0.0520
    10.0000     0.0390
    11.0000     0.0289
```

Option 2: Use the Pascal distribution formula

$$p(k) = \binom{k-1}{r-1} p^r q^{k-r}$$

```
p = 1/3;
q = 2/3;
r = 2;
B = zeros(21,1);
for i=3:20
   k = i-1;
   B(i) = NchooseM(k-1, r-1) * p^r * q^(k-r) .* (k >= 0);
   end

k = [0:20]';
[k(1:21),A2(1:21),B(1:21)]

       k         conv   formula
         0          0         0
    1.0000          0         0
    2.0000     0.1111    0.1111
    3.0000     0.1481    0.1481
    4.0000     0.1481    0.1481
    5.0000     0.1317    0.1317
    6.0000     0.1097    0.1097
```

Option 3: z-transforms

The moment-generating function (i.e. z-transform) for a geometric distribution is

$$\psi(z) = \left(\frac{p}{z-q}\right)$$

Doing this twice gives

$$\psi(z) = \left(\frac{p}{z-q}\right)^2 = \left(\frac{1/3}{z-2/3}\right)^2$$

Take the inverse z-transform. From a table of z-transforms:

$$\left(\frac{z}{(z-a)^2}\right) \leftrightarrow \left(\frac{k}{1!}\right) a^{k-1}$$

so

$$\left(\frac{1/3}{z-2/3}\right)^2 = \left(\frac{1}{9z}\right)\left(\frac{z}{(z-2/3)^2}\right) \rightarrow \left(\frac{1}{9z}\right) k \left(\frac{2}{3}\right)^{k-1} u(k)$$

1/z means delay by one

$$p(k) = \left(\frac{1}{9}\right)\ (k-1)\left(\frac{2}{3}\right)^{k-2}\ u(k-1)$$

Checking in Matlab

```
C = zeros(21,1);
for i=3:21
    k = i-1;
    C(i) = (1/9)*(k-1)* ( (2/3)^(k-2) );
    end

k = [0:20]';
[k(1:21),A2(1:21),B(1:21), C(1:21)]
```

k	conv	formula	z-trans
0	0	0	0
1	0	0	0
2	0.1111	0.1111	0.1111
3	0.1481	0.1481	0.1481
4	0.1481	0.1481	0.1481
5	0.1317	0.1317	0.1317
6	0.1097	0.1097	0.1097
7	0.0878	0.0878	0.0878

Example 2:

Let

- A be the number of times you roll an 8-sided die until you roll a 1 ($p = 1/8$)
- B be the number of times you roll an 8-sided die until you roll a 1 or 2 ($p = 2/8$)
- C be the number of times you roll an 8-sided die until you roll a 1, 2, or 3 ($p = 3/8$)

Determine the pdf for $A + B + C$

Solution using Matlab and Convolutions:

A, B, and C are geometric distributions

$$A(k) = \left(\frac{1}{8}\right)\left(\frac{7}{8}\right)^{k-1} u(k-1)$$

$$B(k) = \left(\frac{2}{8}\right)\left(\frac{6}{8}\right)^{k-1} u(k-1)$$

$$C(k) = \left(\frac{3}{8}\right)\left(\frac{5}{8}\right)^{k-1} u(k-1)$$

Use Matlab to convolve the three together

```
A = (1/8) * (7/8).^(k-1) .* (k-1 >= 0);
B = (2/8) * (6/8).^(k-1) .* (k-1 >= 0);
C = (3/8) * (5/8).^(k-1) .* (k-1 >= 0);
```

k	A	B	C
0	0	0	0
1.0000	0.1250	0.2500	0.3750
2.0000	0.1094	0.1875	0.2344
3.0000	0.0957	0.1406	0.1465
4.0000	0.0837	0.1055	0.0916

Now convolve them

```
AB = conv(A,B);
ABC = conv(AB,C);
[k(1:21),ABC(1:21)]

    k       p(k)
    0          0
    1          0
    2          0
    3     0.0117
    4     0.0264
    5     0.0397
    6     0.0501
    7     0.0572
    8     0.0611
    9     0.0625
   10     0.0619
```

Option 2: z-transforms

$$A(z) = \left(\frac{1/8}{z-7/8}\right)$$

$$B(z) = \left(\frac{2/8}{z-6/8}\right)$$

$$C(z) = \left(\frac{3/8}{z-5/8}\right)$$

The z-transform for the sum of the three is

$$Y(z) = A(z) \cdot B(z) \cdot C(z)$$

$$Y(z) = \left(\frac{1/8}{z-7/8}\right)\left(\frac{2/8}{z-6/8}\right)\left(\frac{3/8}{z-5/8}\right)$$

This isn't in the table of z-transforms, so use partial fraction expansion

$$Y(z) = \left(\frac{1/8}{z-7/8}\right)\left(\frac{2/8}{z-6/8}\right)\left(\frac{3/8}{z-5/8}\right)$$

$$Y(z) = \left(\frac{0.375}{z-7/8}\right) + \left(\frac{-0.75}{z-6/8}\right) + \left(\frac{0.375}{z-5/8}\right)$$

Multiply both sides by z

$$zY = \left(\frac{0.375z}{z-7/8}\right) + \left(\frac{-0.75z}{z-6/8}\right) + \left(\frac{0.375z}{z-5/8}\right)$$

$$z \cdot y(k) = \left(0.375\left(\frac{7}{8}\right)^{k} - 0.75\left(\frac{6}{8}\right)^{k} + 0.375\left(\frac{5}{8}\right)^{k}\right)u(k)$$

Divide by z (delay by 1)

$$y(k) = \left(0.375\left(\frac{7}{8}\right)^{k-1} - 0.75\left(\frac{6}{8}\right)^{k-1} + 0.375\left(\frac{5}{8}\right)^{k-1}\right)u(k-1)$$

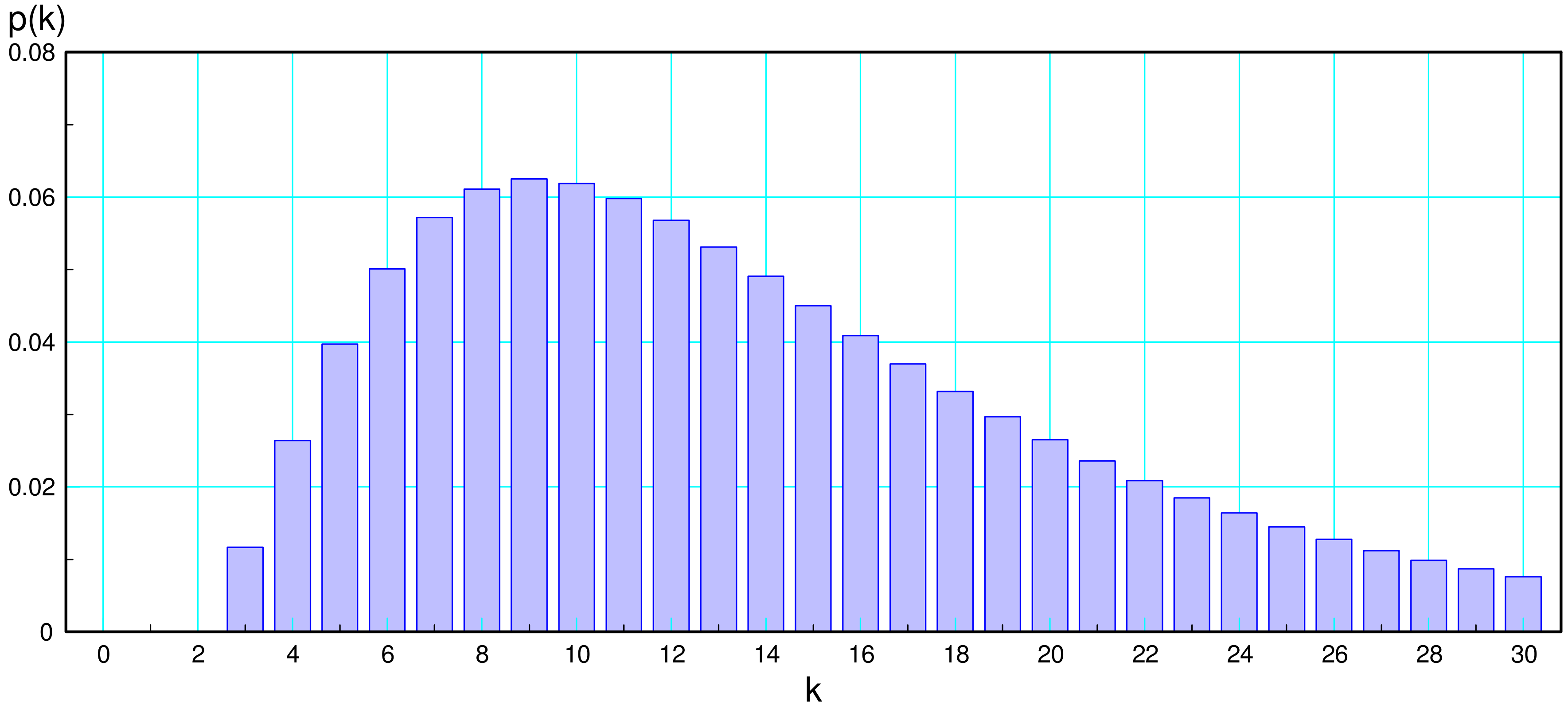
Checking against the results for convolution:

```
k = [0:31]';
Y = 0*k;
Y = ( 0.375*(7/8).^(k-1) - 0.75*(6/8).^(k-1) + 0.375*(5/8).^(k-1) ) .*
(k-1>0);

[k(1:21),ABC(1:21),Y(1:21)]
```

k	conv	z-trans
0	0	0
1	0	0
2	0	0
3	0.0117	0.0117
4	0.0264	0.0264
5	0.0397	0.0397
6	0.0501	0.0501
7	0.0572	0.0572
8	0.0611	0.0611
9	0.0625	0.0625
10	0.0619	0.0619
11	0.0598	0.0598

Either method is valid: they give you the same results.



pdf for the number of rolls of an 8-sided die until you roll a 1, then a 1 or 2, then a 1, 2, or 3