Uniform Distribution

ECE 341: Random Processes

Lecture #7

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Definitions:

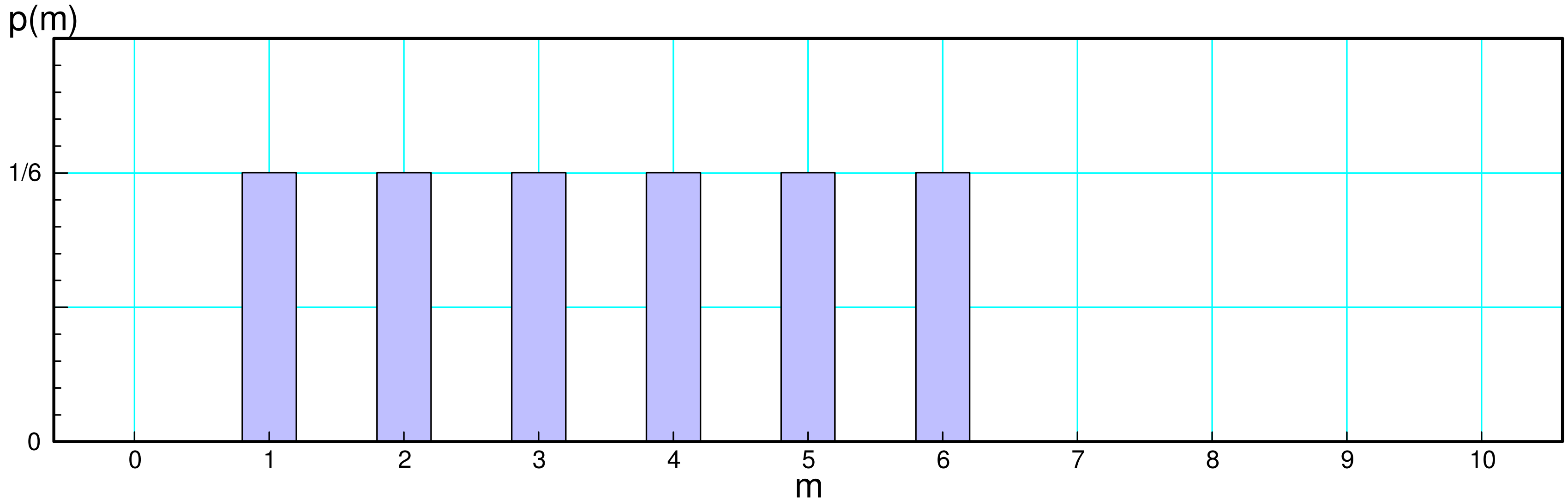
- Uniform Distribution: The probability of each valid outcome is the same.
- Geometric Distribution: The number of Bernoilli trials until you get a success
- Pascal Distribution: The number of Bernolli trials until you get r successes

distribution	description	pdf	mgf	mean	variance
Uniform range = (a,b)	toss an n-sided die	$\left(\frac{1}{1+b-a}\right)\sum_{k=a}^{b}\delta(k-m)$	$\left(\frac{1}{n}\right)\left(\frac{1+z+z^2+\ldots+z^{n-1}}{z^{n-1}}\right)$	$\left(\frac{a+b}{2}\right)$	$\left(\frac{(b+1-a)^2-1}{12}\right)$
Geometric	Bernoulli until 1st success		$\left(\frac{qz}{z-p}\right)$	$\left(\frac{1}{p}\right)$	$\left(\frac{q}{p^2}\right)$
Pascal	Bernoulli until rth success		$\left(\frac{qz}{z-p}\right)^r$	$\left(\frac{r}{p}\right)$	$\left(\frac{qr}{p^2}\right)$

Uniform Distribution:

A Bernoulli trial where

- There are n possible outcomes (rather than just two), and
- All outcomes have the same probability.



pdf for a uniform distribution (6-sided die)

Examples:

- Drawing a card from a deck of cards (each has a probability of 1/52)
- A number coming up in Roulette (1 in 31 in Vegas, 1 in 32 in Atlantic City)
- A number coming up in the lottery (1 in 78,960,960)
- Being selected for jury duty (1 in 15,000. Ten people are selected from a county population of 150,000)

There are some betting schemes which take advantage of processes which are supposed to be uniform but are not. For example, about 15 years ago, someone watched which numbers came up in Roulette in Vegas and found that some numbers were more common that others. He/she (I forget which) won money with this scheme. Now, the roulette wheels are mixed every night (under tight security) and you are not allowed to watch and take notes.

Properties of a Uniform Distribution: (a,b)

pdf: $X(m) = \left(\frac{1}{n}\right)\sum_{k=a}^{b} \delta(k-m)$

mgf $\psi(z) = \left(\frac{1}{b-a}\right)\left(\frac{1}{z^a} + \frac{1}{z^{a+1}} + \ldots + \frac{1}{z^b}\right)$

Mean: $\mu = \left(\frac{a+b}{2}\right)$

Var $\sigma^2 \approx \left(\frac{(b-a)^2}{12}\right)$

note: If you add two distributions

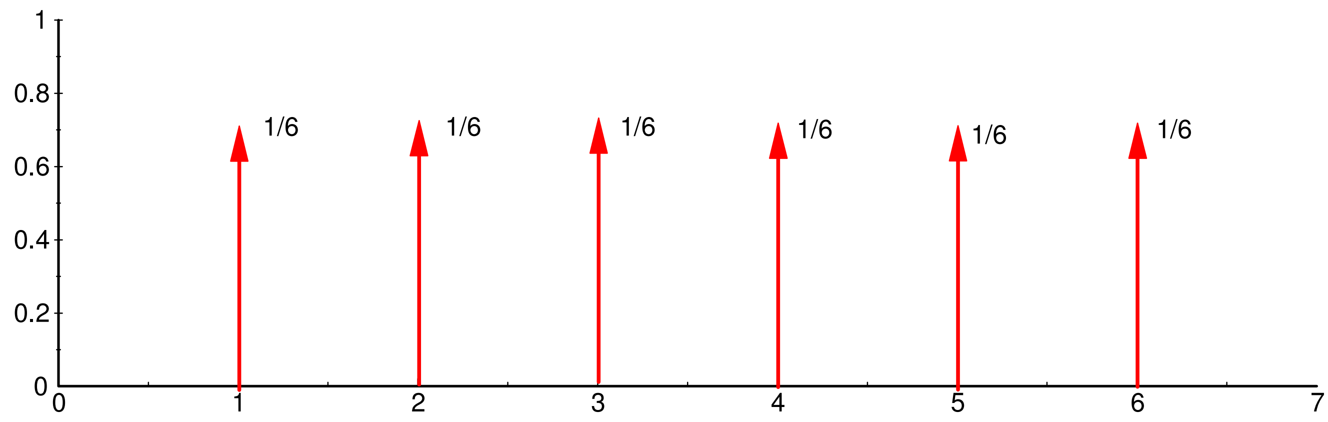
- The means add
- The variances add

pdf of one 6-Sided Die (1d6)

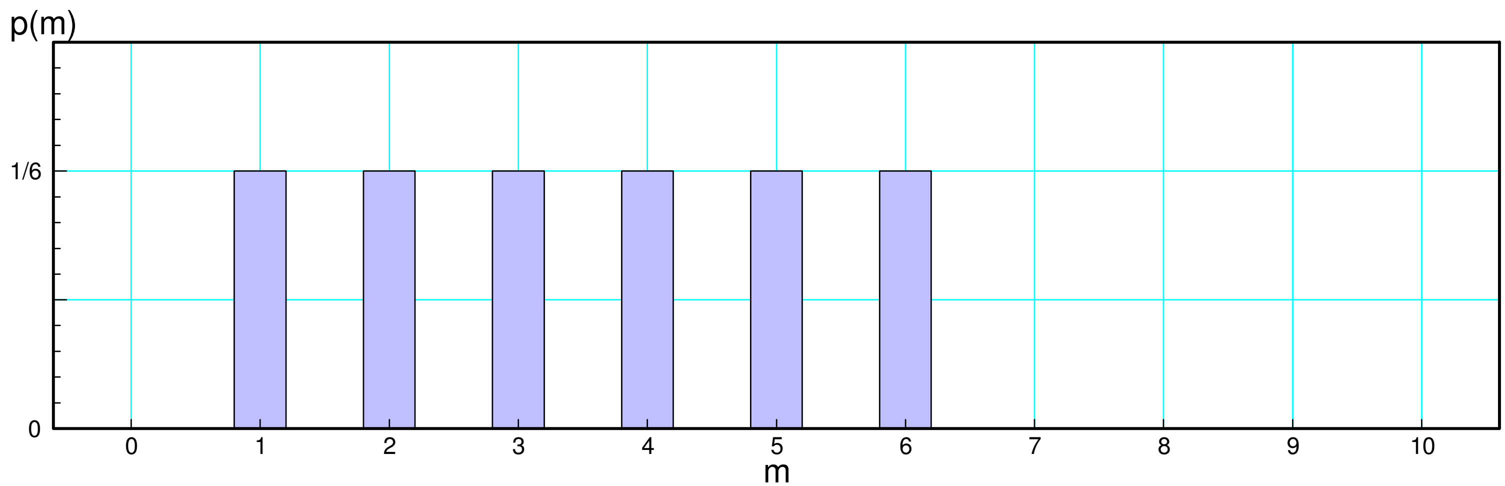
For example, a fair six sided die would have six possible outcomes, each with probability of 1/6

Die Roll: x	0	1	2	3	4	5	6	7+
p(x)	0	1/6	1/6	1/6	1/6	1/6	1/6	0

The pdf for this would be a delta function at each integer value:



pdf for a fair 6-sided die. Can be representated witha bar graph too



Bar graph representation for the pdf of a 6-sided die

Mean: $\mu = \left(\frac{1+6}{2}\right) = 3.5$

Var: $\sigma^2 = \left(\frac{(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)}{6}\right) = 2.91167$

pdf for two 6-sided die (2d6)

Option 1: Comninatorics

```
0:  -
1:  -
2:  (1,1)
3:  (1,2), (2,1)
4:  (1,3), (2,2), (3,1)
5:  (1,4), (2,3), (3,2), (4,1)
6:  (1,5), (2,4), (3,3), (4,2), (5,1)
7:  (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)
8:  (2,6), (3,5), (4,4), (5,3), (6,2)
9:  (3,6), (4,5), (5,4), (6,3)
10: (4,6), (5,5), (6,4)
11: (5,6), (6,5)
12: (6,6)
13: -
14: -
```

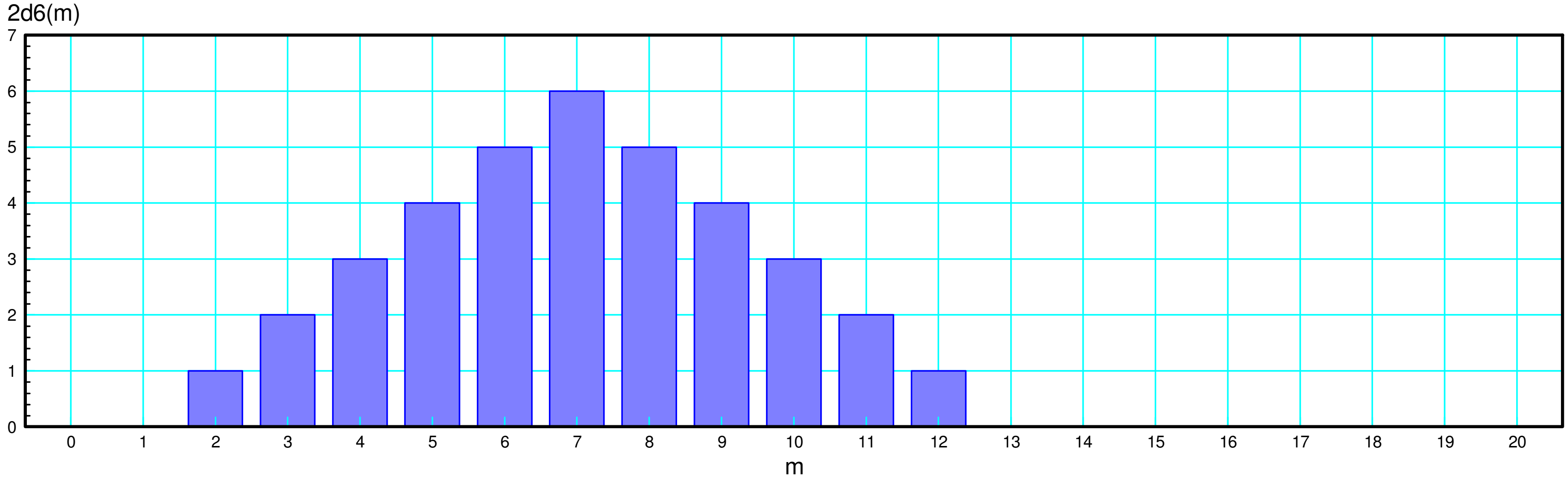
Option 2: Convolution

- pdf for 1d6 (scaled by 6)

```
d6 = [0,1,1,1,1,1,1]
```

- pdf for 2d6 (scaled by 6^2)

```
d6x2 = conv(d6,d6)
d6x2 =  0   0   1   2   3   4   5   6   5   4   3   2   1
```



pdf of rolling 2d6 (scaled by 36)

mean and variance for 2d6:

$$\mu = 2 \cdot 3.5 = 7$$

$$\sigma^2 = 2 \cdot 2.9167 = 5.8333$$

Compute the mean and variance in Matlab

```
>> d6 = [0 1 1 1 1 1 1]' /6;
>> d6x2 = conv(d6,d6);
>> k = [0:12]';
>>
>> % mean
>> x = sum(d6x2 .* k)

x =      7

>> % variance
>> s2 = sum( (k-x).^2 .* d6x2 )

s2 =    5.8333
```

Check with a Monte Carlo simulation

```
for i=1:10
   DICE = ceil(6*rand(1,2));
   disp([DICE, sum(DICE)])
end
```

A	B	A+B
5	1	6
2	2	4
5	1	6
4	6	10
5	2	7
1	5	6
4	4	8
4	3	7

Now place the sum into an array and take the mean and standard deviation

```
X = zeros(1e5, 1);

for i=1:1e5
   DICE = ceil(6*rand(1,2));
   X(i) = sum(DICE);
end

x = mean(X)
s2 = std(X)^2


x =    7.0053     vs. 7.0000

s2 =   5.8375     vs. 5.8333
```

Problem: Determine the probability of rolling 10 or higher with 2d6.

Solution: The number of combinations that result in rolls of 10 or higher are

- 10: frequency = 3
- 11: frequency = 2
- 12: frequency = 1

so the probability is

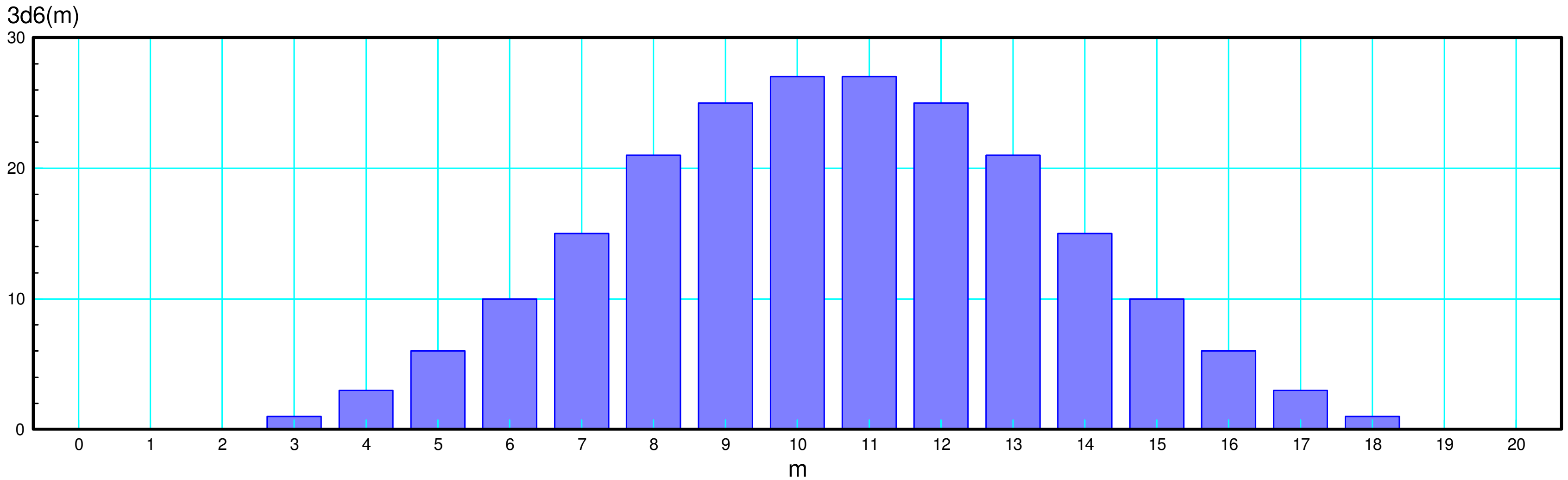
$$p(m \geq 10) = \left(\frac{3+2+1}{36}\right) = \frac{6}{6}$$

Three six-sided dice (3d6)

Scaling the pdf of a 6-sided die by 6 results in the pdf scaled by 6^3 (216)

```
d6x3 = conv(d6x2, d6)
```

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	0	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1



pdf of rolling 3d6 (scaled by 216)

The probability of rolling 16 or higher is thus

$$p(m \geq 16) = \left(\frac{6+3+1}{216}\right) = \left(\frac{10}{216}\right)$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	**16**	**17**	**18**
0	0	0	1	3	6	10	15	21	25	27	27	25	21	15	10	**6**	**3**	**1**

The mean is

$$\mu = 3 \cdot 3.5 = 10.5$$

The variance is

$$\sigma^2 = 3 \cdot 2.9167 = 8.75$$

Checking in Matlab (take 1)

```
>> d6 = [0 1 1 1 1 1 1]' /6;
>> d6x2 = conv(d6,d6);
>> d6x3 = conv(d6x2,d6);
>> k = [0:18]';
>> x = sum(d6x3 .* k)

x =   10.5000

>> s2 = sum( (k-x).^2 .* d6x3 )

s2 =    8.7500
```

Checking in Matlab (take 2)

- Monte-Carlo simulation with 100,000 trials

Start with 10 trials:

```
for i=1:10
   DICE = ceil(6*rand(1,3));
   disp([DICE, sum(DICE)])
end
```

A	B	C	A+B+C
4	4	2	10
4	6	2	12
1	3	3	7
6	3	5	14
2	5	3	10
4	5	3	12
5	4	5	14
5	2	3	10
2	3	5	10
3	5	5	13

Now roll 100,000 times:

```
X = zeros(1e5, 1);

for i=1:1e5
   DICE = ceil(6*rand(1,3));
   X(i) = sum(DICE);
end

x = mean(X)
s2 = std(X)^2
```

resulting in

```
       exp        calculated
x =    10.5015    10.50

s2 =   8.6896     8.75
```

Final Problem:

- Find the pdf for Y = d4 + d6 + d8 + d10

Solution: Use convolution

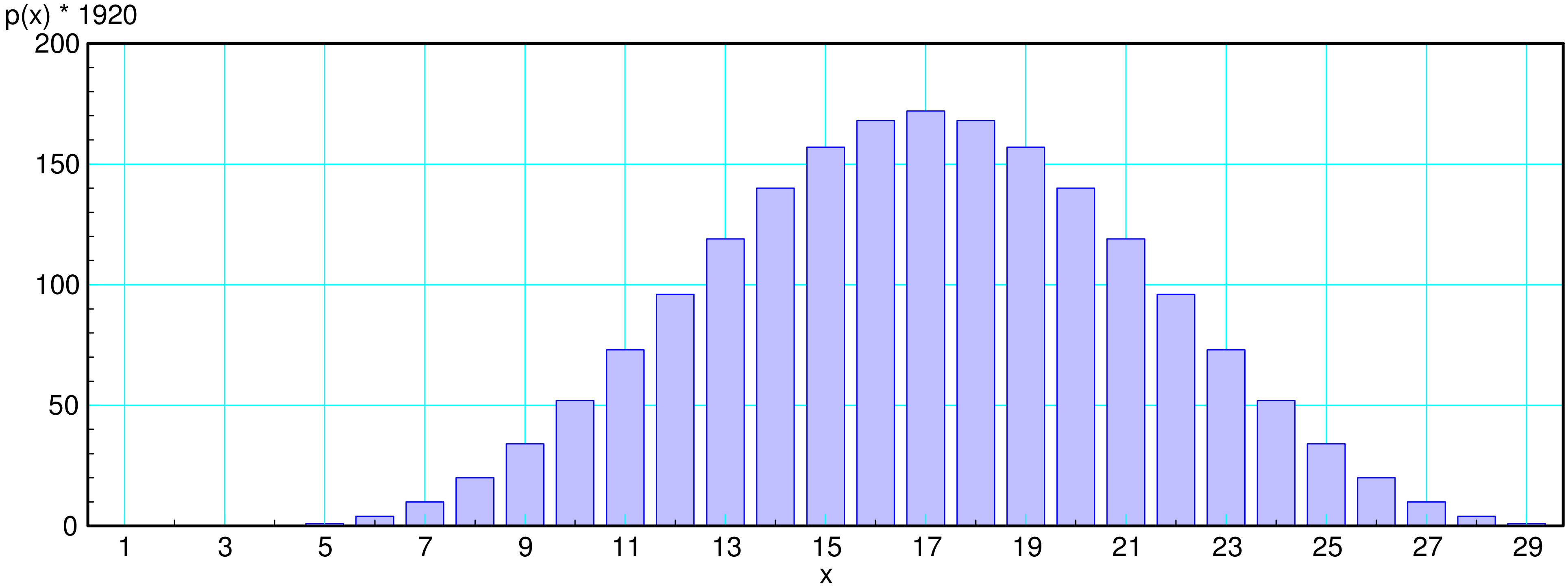
```
d4 = [0, 1, 1, 1, 1];
d6 = [0, 1, 1, 1, 1, 1, 1];
d8 = [0, 1, 1, 1, 1, 1, 1, 1, 1];
d10 = [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1];

a = conv(d4,d6);
b = conv(d8,d10);
Y = conv(a, b)

sum(Y)
        1920

There are 1920 different combinations
The probabilities are Y / 1920

bar(Y)
```



pdf for the sum: Y = d4 + d6 + d8 + d10

Note that the resulting pdf is a bell shaped curve (central limit theorem)