Bernoulli Trials & Binomial Distributions

ECE 341: Random Processes

Lecture #6

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Definitions:

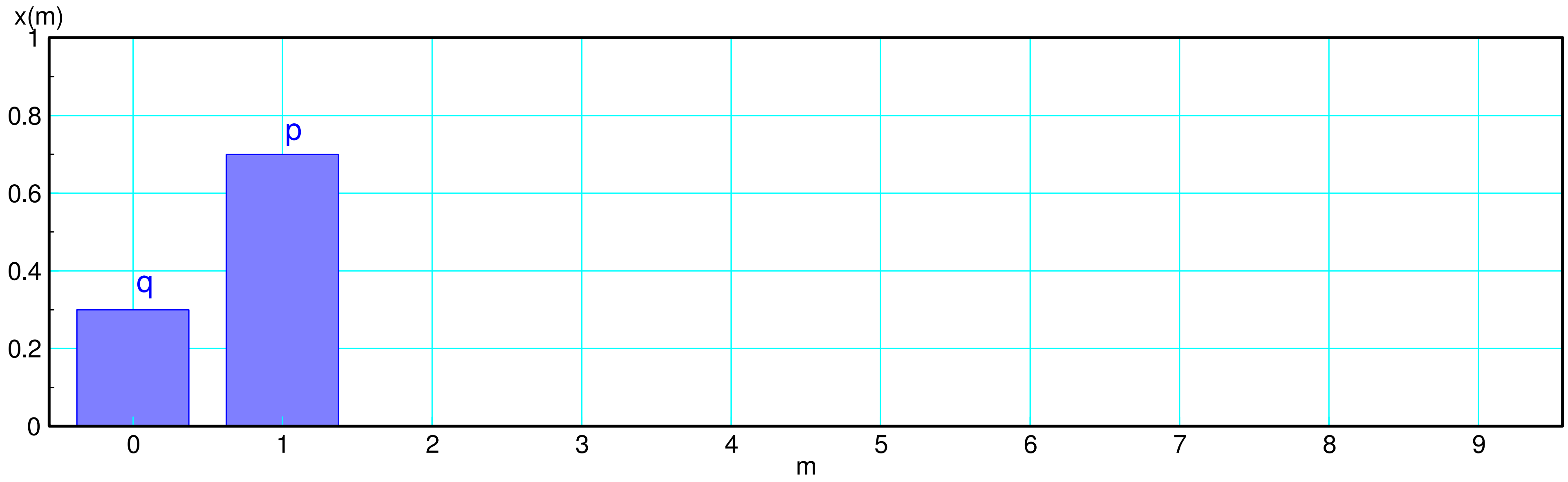
- Bernoulli Trial: A random event whose outcome is true (1) or false (0).
- Binomial Distribution: n Bernoulli trials.
- p — The probability of a true (1) outcome (also called a success)
- q — 1-p. The probability of a false (0) outcome (also called a failure)
- n — The number of trials
- $f(x|n,p)$ — The probability density function with sample size n and probability of success p.

distribution	description	pdf	mgf	mean	variance
Bernoulli trial	flip a coin obtain m heads	$p^m q^{1-m}$	$q+\frac{p}{z}$	p	p(1-p)
Binomial	flip n coins obtain m heads	$\binom{n}{m} p^m q^{n-m}$	$\left(q+\frac{p}{z}\right)^n$	np	np(1-p)
Hyper Geometric	Bernoulli trial without replacement	$\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$			

Bernoulli Trial:

A Bernoulli trial is essentially a coin flip:

- There are only two possible outcomes (heads or tails, 1 or 0, success or failure)
- The probability of a success is p
- The probability of a failure is q (which must be 1-p for probabilities to add to 1.000)



probability density function for a Bernoulli trial. For illustration, q = 0.3, p = 0.7

With only two possible outcomes, there are only two bars. You can also write this as

$$x(m) = q \cdot \delta(m) + p \cdot \delta(m-1)$$

Note that the mathematics doesn't care what integer *m* represents:

- In ECE 343 Signals and Systems, m represents time
- In ECE 341 Random Processes, m represents the number of successes

The math doesn't care: it's just an integer, m. Likewise, the same math used in Signals and Systems (i.e. the z-transform) applies to Random Processes.

The z-operator is defined as

$$zx(m) \equiv x(m+1)$$

where *zX* is read as *the next value of x*. If multiplying by z moved forward in time, then dividing by z moves backwards in time

$$z^{-1}x(m) \equiv x(m-1)$$

Likewise, the z-transform for a Bernoulli trial (termed the *moment generating function*) is

$$\psi(z) = q + z^{-1}p = \left(\frac{qz+p}{z}\right)$$

Mean

- The mean is the average.
- Your average winnings from a coin toss are

$$\mu = mean$$

$$\mu = \sum k \cdot p(k)$$

$$\mu = (0 \cdot q) + (1 \cdot p)$$

$$\mu = p$$

If you get $1 if you win, you expect to make $p every time you flip the coin.

Variance & Standard Deviation

The variance and standard deviation is a measure of the variability

$$\sigma^2 = variance$$

$$\sigma = standard\ deviation$$

These are defined as

$$\sigma^2 = \sum p(k) \cdot (k - \bar{x})^2$$

For a Bernoulli trial

$$\sigma^2 = q(0 - p)^2 + p(1 - p)^2$$

$$\sigma^2 = p(1 - p)$$

Bernoulli Trials in Matlab:

Matlab has random number functions:

rand	*generate a random number in the interval of 0..1*
rand(10,1)	*generate a matrix of 10 random numbers over the range of 0..1*
randn	*generate a random number with a Gaussian distribution*

For a Bernoulli trial, use the *rand* function

- randn uses a standard normal distribution - covered later in this course

Flip a coin 5 times with the probability of a heads being 0.7.

- First, generate 5 random numbers in the interval (0,1):

```
X = rand(5,1)

    0.8147
    0.9058
    0.1270
    0.9134
    0.6324
```

- Next, convert this to a binary (1 or 0) number

```
Coin = 1 * (rand(5,1) < 0.7)

     0
     0
     1
     0
     1
```

Flip 100 coins

- Find the mean and standard deviation

Solution: Flip a coin 100 times

```
Coin = 1 * (rand(100,1) < 0.7);
x = mean(Coin)

x =   0.5900      vs. 0.700 expected

s = std(Coin)

s =   0.4943      vs 0.4528 expected
```

Flip a coin 1 million times,

```
Coin = 1 * (rand(1e6,1) < 0.7);
x = mean(Coin)

x =    0.6997        vs. 0.7000 expected

s = std(Coin)

s =    0.4584        vs. 0.4528 expected
```

That is essentially the definition of probability: as the number of coin flips goes to infinity, the number of successes divide by the number of trials approaches p

$$\lim_{trials \to \infty} \left(\frac{\text{\# successes}}{\text{\# trials}} \right) = p$$

Sidelight: What is the probability of the Vikings winning their home opener?

- *This is not a repeatable event*
- *Probability does not properly apply*

Actual Question: What is the betting line on the Vikings winning their home opener?

- *Ideally, the losers pay the winners*
- *The house wins on ties*

Binomial Distribution

- The binomial distribution is the sum of n Bernoulli trials.
- The probability of getting m heads with n trials is

$$X(m) = \binom{n}{m} p^m q^{n-m}$$

There are several ways to get this result.

Enumeration

Question: What is the probability of getting m heads in 2 trials?

Answer 1: Enumerate all possibilities and probabilities:

result	probability
1 1	p * p
1 0	p * q
0 1	q * p
0 0	q * q

or

m	X(m)
2	p^2
1	2pq
0	q^2

Problem: What is the probability of flipping m heads in 3 trials (n=3)

Enumerating all combinations (with 1 meaning a heads (success))

n	combinations	probability
3	111	p^3
2	110, 101, 011	$3p^2q$
1	001, 010, 001	$3pq^2$
0	000	q^3

Convolution

The probability density function for a Bernoulli trial is

$$x(m) = q \cdot \delta(m) + p \cdot \delta(m-1)$$

Another way to write this is

$$x(m) = \sum_k x(k) \cdot \delta(m-k)$$

This works since

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$$

Likewise, the delta function pick out only those values of x(k) when k = m, giving

$$x(m) = x(m)$$

That seems kind of silly, but it helps when dealing with two coin tosses. If you are tossing two coins, the resulting pdf is

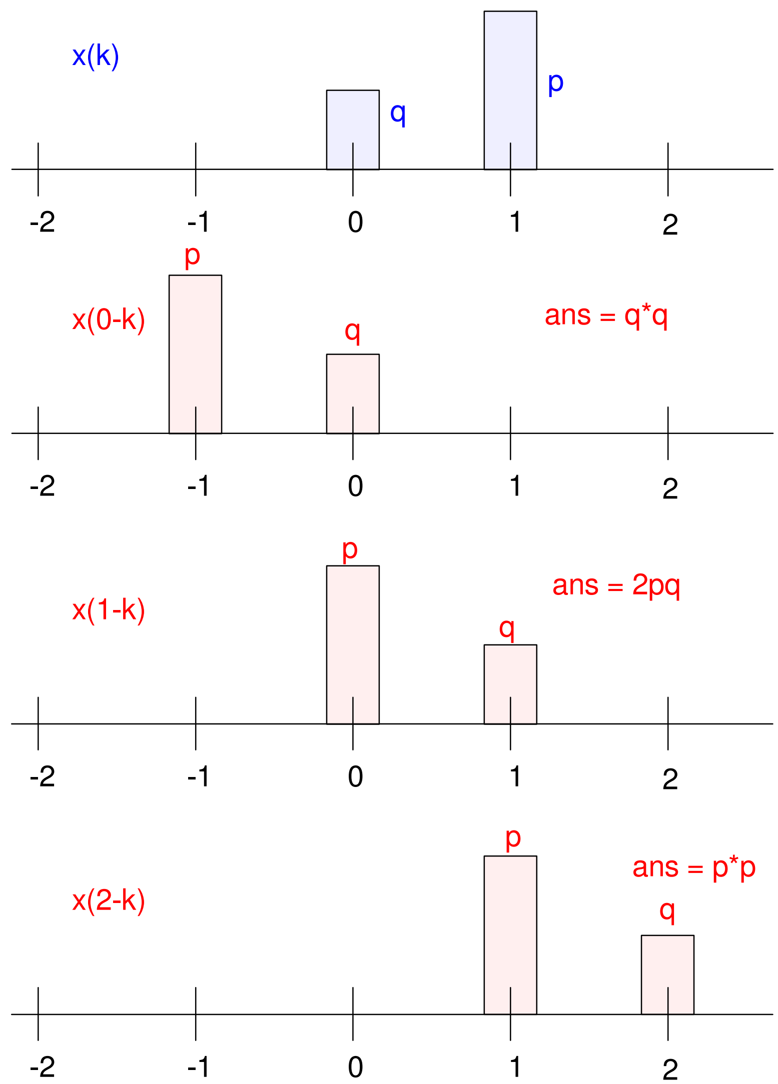
$$y(m) = \sum_k x(k)x(m-k)$$

or

$$y(m) = x(m) ** x(m)$$

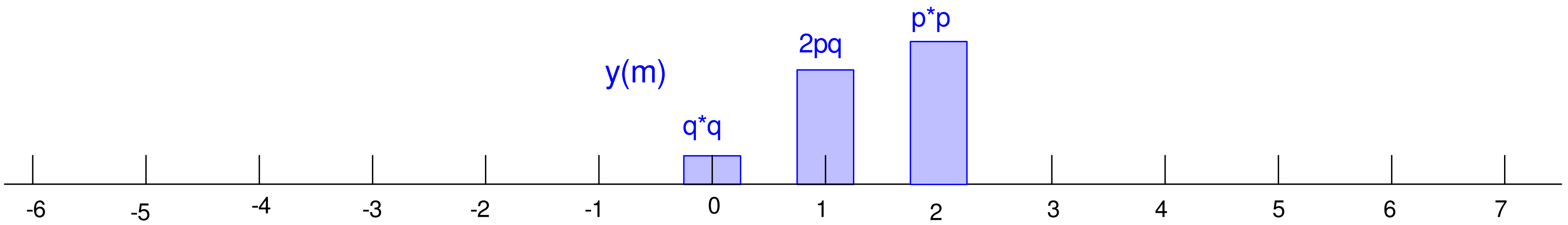
where ** denotes convolution. Graphically, what you are doing is

- Flip x(m) to get x(0-m)
- Multiply x(k) with x(-k) and sum the results. This gives $y(0) = q^2$



Result

$$y(m) = q^2 \cdot \delta(m) + 2qp \cdot \delta(m-1) + p^2 \cdot \delta(m-2)$$



In Matlab, this is the convolution function

```
x = [0.3,0.7]

        x(0)        x(1)
x =    0.3000      0.7000

y = conv(x,x)

         y(0)        y(1)         y(2)
y =    0.0900      0.4200      0.4900
```

z-Transform

A property of the z-transform is

- Convolution in the time domain (probability domain here) is multiplication in the z-domain

This is essentially because multiplying polynomials is convolution.

Example: Find the product of

$$C = AB = (2x^2 + 3x + 4)(5x^3 + 6x^2 + 7x + 8)$$

- Solution using convolution: Flip B(x) and start shifting. The C(0) = 32

B(k)	0	0	0	**8**	7	6	5	0	0	0
A(-k)	0	2	3	**4**	0	0	0	0	0	0
A*B	0	0	0	**32**	0	0	0	0	0	0

Shift A by one and multiply. C(1) = 24 + 28 = 52

B(k)	0	0	0	**8**	**7**	6	5	0	0	0
A(1-k)	0	0	2	**3**	**4**	0	0	0	0	0
A*B	0	0	0	**24**	**28**	0	0	0	0	0

Shift A by one and multiply. C(2) = 16 + 21 + 24 = 61

B(k)	0	0	0	**8**	**7**	**6**	5	0	0	0
A(2-k)	0	0	0	**2**	**3**	**4**	0	0	0	0
A*B	0	0	0	**16**	**21**	**24**	0	0	0	0

Shift A by one and multiply. C(3) = 14 + 18 + 20 = 52

B(k)	0	0	0	8	**7**	**6**	**5**	0	0	0
A(3-k)	0	0	0	0	**2**	**3**	**4**	0	0	0
A*B	0	0	0	0	**14**	**18**	**20**	0	0	0

This is the *conv* operation in Matlab:

```
A = [4,3,2];
B = [8,7,6,5];
C = conv(A,B)

         x0     x1      x2     x3     x4     x5
C =      32     52     61     52     27     10
```

$$(4+3x+2x^2)(8+7x+6x^2+5x^3)$$

$$=32+52x+61x^2+52x^3+27x^4+10x^5$$

The z-transform of two Bernoulli trials is thus

$$Y(z) = (q + z^{-1}p)(q + z^{-1}p)$$

$$Y(z) = q^2 + 2pqz^{-1} + p^2 z^{-2}$$

Problem: Determine the probability distribution of flipping 6 coins.

Solution 1: Convolve 6 times

```
X = [0.3, 0.7]
X =    0.3000    0.7000
Y2 = conv(X,X)
Y2 =    0.0900    0.4200    0.4900
Y3 = conv(Y2,X)
Y3 =    0.0270    0.1890    0.4410    0.3430
Y4 = conv(Y3,X)
Y4 =    0.0081    0.0756    0.2646    0.4116    0.2401
Y5 = conv(Y4,X)
Y5 =    0.0024    0.0284    0.1323    0.3087    0.3601    0.1681
Y6 = conv(Y5,X)
Y6 =    0.0007    0.0102    0.0595    0.1852    0.3241    0.3025    0.1176
```

Solution 2: The probability of m heads in n tosses is

$$y(m) = \binom{n}{m} p^m q^{n-m}$$

For example, the probability of 4 heads is

$$y(4) = \binom{6}{4} (0.7)^4 (0.3)^2$$

$$y(4) = 0.324135$$

which matches the Matlab results

```
Y6 = conv(Y5,X)
Y6 =    0.0007    0.0102    0.0595    0.1852    0.3241    0.3025    0.1176
```

Pascal's Triangle

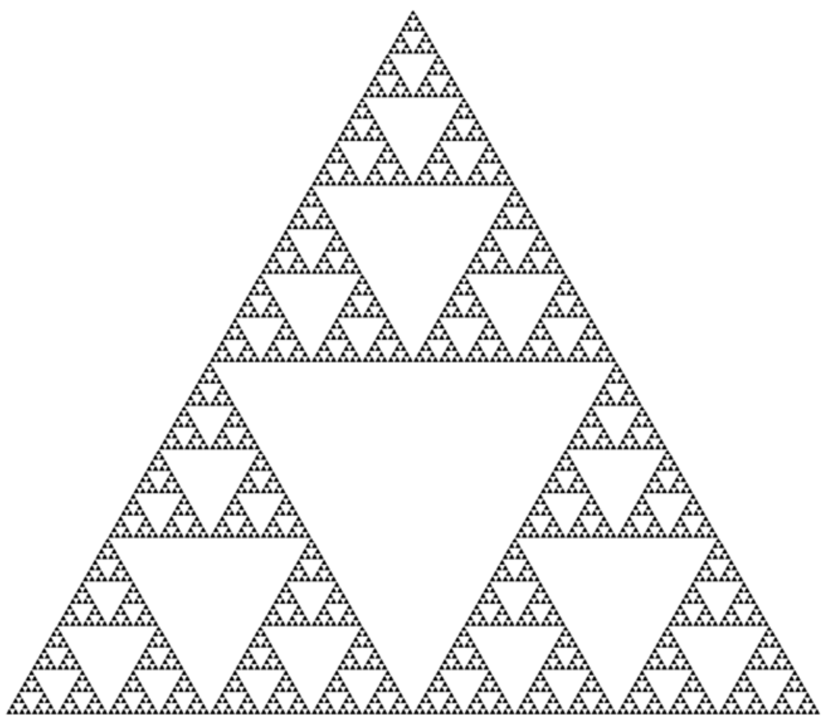
A kind of neat pattern for the combination $\binom{n}{m}$ is as follows:

- Start with the number 1 (or 0 1 0) in row #1
- Offset row #2 by 1/2 a digit. Add the numbers to the left and right of each spot in row #1 to generate row 2.
- Offset row #3 by 1/2 a digit. Add the numbers to the left and right of each spot in row #2 to generate row 3.

The result is as follows:

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
   1     5    10    10     5     1
1     6    15    20    15     6     1
```

Each row is the value of $\binom{n}{m}$ as m goes from zero to n - which is one way of computing combinatorics. If you shade in the odd entries, you get a pretty picture as well: (in the limit becoming Sierpinski Triangle)



SierpinskiTriangle / Pascal's Triangle - source Wikipedia.com

Central Limit Theorem (take 1)

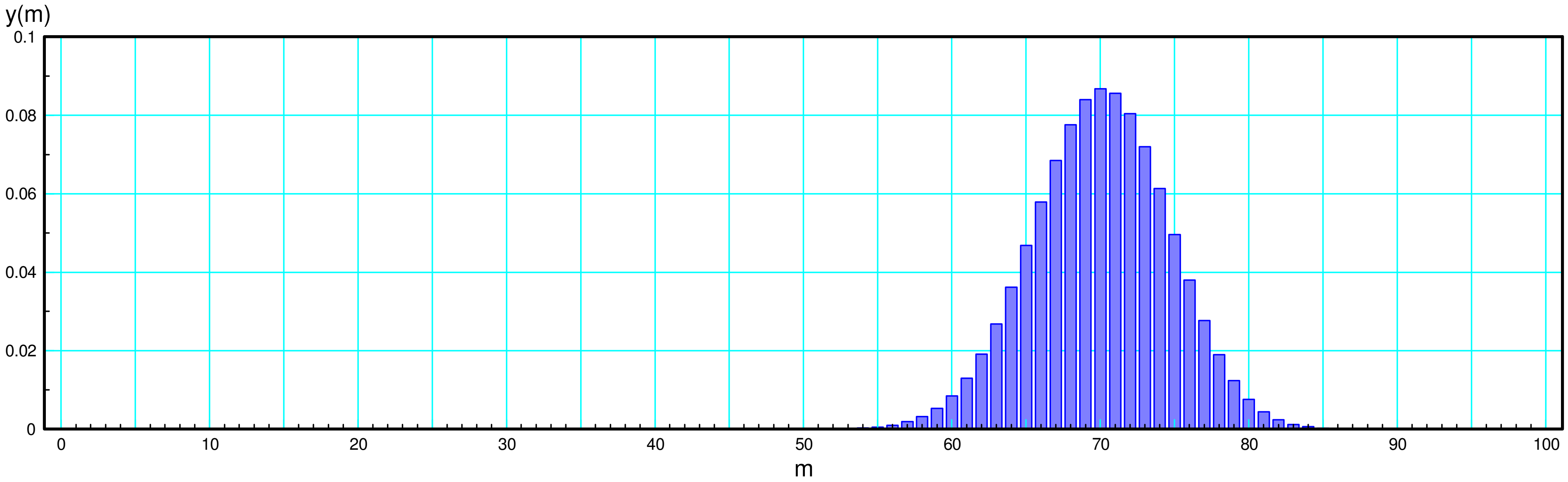
Determine the pdf of flipping 100 coins.

Solution (use Matlab for this):

$$y(m) = \binom{100}{m} p^m q^{100-m}$$

In Matlab

```
Y = zeros(101,1);
for m=0:100
   Y(m+1) = factorial(100) / (factorial(m) * factorial(100-m)) * 0.7^m  * 0.3^(100-m);
end
bar(Y)
```



Note that this is a bell-shaped curve, called a *Normal Distribution*.

- The Central Limit Theorem states that all distributions converge to this shape.
- It's probably the most important distribution in statistics - we'll be covering it later.

In addition, all probabilities add to one:

```
sum(Y)

ans =   1.0000
```

The mean of a binomial distribution is np (70)

```
sum(Y .* m)

ans =   70.0000
```

The variance is npq (21)

```
sum(Y .* (m - 70).^2)

ans =   21.0000
```

Binomial Distributions with Multiple Outcomes:

Problem: Two people are playing tennis.

- p(A) winning is 60%
- Best of 7 series (first person to win 4 games wins the match)

p(A Winning) = p(A=4) + p(A=5) + p(A=6) + p(A=7)

4 wins: $$f(4) = \binom{7}{4}(0.6)^4(0.4)^3 = 0.2903$$

5 wins: $$f(5) = \binom{7}{5}(0.6)^5(0.4)^2 = 0.2613$$

6 wins $$f(6) = \binom{7}{6}(0.6)^6(0.4)^1 = 0.1306$$

7 wins: $$f(7) = \binom{7}{7}(0.6)^7(0.4)^0 = 0.0280$$

The total is 0.7102.

Problem: Two people are playing tennis.

- p(A) winning is 60%
- First one to be **up** 4 games wins the match

What is the chance that A wins the match?

Solution: This is a *totally* different problem. If person A wins followed by a loss, you're back where you started. The net result is potentially an infinite series. This is a Markov chain (coming later this semester).

Hypergeometric Distribution:

Problem: Suppose a box contains A white balls and B black balls. Each trial, you take one ball out of the bin and then put it back into the bin. Find the probability distribution function for drawing n white balls.

Solution: This is sampling with replacement. Each trial has the same probability of success (drawing a white ball)

$$p = \frac{A}{A+B}$$

The probability of n white balls is then a binomial distribution

$$f(x|n,p) = \binom{n}{x} p^x q^{n-x}$$

Problem: Suppose you do *not* replace the ball after you select it. This changes the problem considerably.

First, you cannot draw more white balls than there are balls in the bin and you can't draw more white balls than the total number of balls you draw:

$$x \le \min(n, A)$$

You also cant draw less than zero balls:

$$x \ge \max(0, n - B)$$

The probability of drawing x balls in n draws is from the following:

- There are $\binom{A}{x}$ ways of drawing x white balls
- There are $\binom{B}{n-x}$ ways of drawing the remaining n-x black balls
- The total number of ways you can draw n balls is $\binom{A+B}{n}$

So, the pdf for a Hypergeometric distribution is:

$$f(x|A,B,n) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$$

Example: A bin has 8 white balls and 10 black balls. You draw 5 balls without replacement. Find the probability that three of the balls are white:

$$f(3|8,10,6) = \frac{\binom{8}{3}\binom{10}{2}}{\binom{18}{5}} = \frac{(56)(45)}{8568} = 0.2941$$