Conditional Probability

5-Card Draw

ECE 341: Random Processes

Lecture #3

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

5-Card Draw

Each player is dealt 5 cards

- 1st round of betting

Each player can dicard 0-5 cards

- Draw the same number of cards
- Results in a 5-card hand
- 2nd round of betting

Players reveal their hands

- Best hand wins

What are the odds of each hand?

Enumeration is now very difficult

- Up to 10 cards drawn per person
- $\binom{52}{10} = 15,829,924,220$ different combinations

Further complicated by the number of cards drawn

- If you have a straight, a flush, or a full house, you'll draw zero cards
- If you have 3 of a kind, you might draw two cards
- If you have a pair, you might draw three cards
- If you have junk, you might draw five cards

This makes computations very difficult

- Use conditional probabilities

Conditional Probabilities (review)

$$p(A) = p(A|B)p(B) + p(A|C)p(C) + ...$$

Example:

- Player A rolls one die
- Player B rolls two dice and takes the maximum
- The highest number wins. A wins on ties.

What is the probability that A wins?

- 3 dice = 6^3 = 216 permutations
- Treat as a conditional probability

Step 1: Compute the p(B)

- 2 dice take the maximum

By enumeration

B	1	2	3	4	5	6
p(B)	1/36	3/36	5/36	7/36	9/36	11/36

B's Score

	1st Die					
2nd Die	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Step 2: Compute the conditional probabilities

A wins

	A's Roll 1	2	3	4	5	6	p(B)
B's Roll 1	1	1	1	1	1	1	1/36
2	0	1	1	1	1	1	3/36
3	0	0	1	1	1	1	5/36
4	0	0	0	1	1	1	7/36
5	0	0	0	0	1	1	9/36
6	0	0	0	0	0	1	11/36

- $p(A) = p(A|B=1)p(B=1) + p(A|B=2)p(B=2) + p(A|B=3)p(B=3) + p(A|B=4)p(B=4) \ldots$
- $p(A) = (6/6)(1/36) + (5/6)(3/36) + (4/6)(5/36) + (3/6)(7/36) + (2/6)(9/36)+(1/6)(11/36)$
- $p(A) = 91/216$

A has a 42.13% chance of winning this game

Draw Poker Odds

Treat as a conidtional probability

- p(4 of a kind) = p(4 of a kind | dealt 4 of a kind) p(dealt 4 of a kind)
- + p(4 of a kind | dealt 3 of a kind) p(dealt 3 of a kind)
- + p(4 of a kind | dealy pair) p(dealt a pair)
- + p(4 of a kind | dealy high card) p(dealt high card)

Repeat for all of the hand types

What we know

- From previous lecture
- p(B) for conditional probabilities

POKER HAND		COUNT	Odds Against
2	Straight Flush	40	64,974
3	Four of a Kind	624	4,165
4	Full House	3,744	694.17
5	Flush	5,108	508.8
6	Straight	10,200	254.8
7	Three of a Kind	54,912	47.33
8	Two Pair	123,552	21.04
9	One Pair	1,098,240	2.37
10	High Card	1,302,540	2
Total		2,598,960	

Conditional Probabilities

- Assume you are dealt a hand, such as 3 of a kind
- Compute the odds of ending up with a royal flush (etc) after drawing 2 cards
- Repeat for each type of hand

Assume a fixed rule for the number of cards drawn

- Ignores attempts to draw to a flush or straight

Poker hand	probability	# cards drawn
Straight Flush	40	0
Four of a Kind	624	0
Full House	3,744	0
Flush	5,108	0
Straight	10,200	0
Three of a Kind	54,912	2
Two Pair	123,552	1
One Pair	1,098,240	3
High Card	1,302,540	5
All Hands	2,598,960	

Four-of-a-Kind:

Assume there are four ways you can end up with a 4-of-a-kind:

Start with

- A: 4-of-a-kind (and do nothing)
- B: 3-of-a-kind (draw 2 new cards)
- C: Pair (draw 3 new cards)
- D: High-Card hand (draw 5 new cards)

The probability of getting a 4-of-a-kind is

$$p(x) = P(x|A)p(A) + p(x|B)p(B) + p(x|C)p(C) + p(x|D)p(D)$$

A: Starting with 4-of-a-kind

The probability of ending up with a 4-of-a-kind is 1.000

$$p(x|A) = 1.000$$

B: Starting with 3-of-a-kind (and draw 2 cards)

The number of ways you can draw 2 cards is

$$M = \binom{47}{2} = 1,081$$

The number of ways you can end up with a 4-of-a-kind is

- Of the one remaining card that matches the cards you keep, choose 1 (1 choose 1)
- Of the 46 remaining cards in the deck, choose 1 (46 choose 1)

$$N = \binom{1}{1}\binom{46}{1} = 46$$

The odds are then

$$p(x|B) = \left(\frac{46}{1081}\right) = \frac{1}{23.5}$$

C: Starting with a pair and you draw 3 new cards

The number of ways you can draw 3 new cards is

$$M = \binom{47}{3} = 16,215$$

To end up with a 4-of-a-kind

- Of the 2 matching cards in the deck, pick two
- Of the remaining 45 cards, pick one

$$N = \binom{2}{2}\binom{45}{1} = 45$$

The odds of getting a four-of-a-kind is then

$$p(x|C) = \left(\frac{45}{16,215}\right) = \left(\frac{1}{360.33}\right)$$

D: High-Card Hand. Draw 4 new cards.

The number of ways to draw 4 cards

$$M = \binom{47}{4} = 178,365$$

The number of ways to get 4 cards that match the one you kept

$$N = \binom{3}{3}\binom{44}{1} = 44$$

The number of ways that the 4 cards you drew are all the same

$$N = 8$$

The odds are then

$$p = \tfrac{44+8}{178,365} = \left(\tfrac{1}{3430}\right)$$

Result: The probability of getting 4-of-a-kind with 5-card draw is thus

p(x) = P(x|A)p(A) + p(x|B)p(B) + p(x|C)p(C) + p(x|D)p(D)

$$p(x) = (1.000)\left(\frac{1}{4165}\right)$$ *dealt 4-of-a-kind*

$$+\left(\frac{1}{23.5}\right)\left(\frac{1}{47.33}\right)$$ *dealt 3-of-a-kind*

$$+\left(\frac{1}{360.33}\right)\left(\frac{1}{2.37}\right)$$ *pair*

$$+\left(\frac{1}{3430}\right)\left(\frac{1}{2}\right)$$ *high-card hand*

$$p(x) = 0.002465 = \frac{1}{407.17}$$

Monte-Carlo Simulation

Repeat the previous program for 5-card draw. Add a block of code which

- Sorts the cards based upon their frequency (plus 0.1 * suit so the sort gives unique results)
- Keeps all cards that pair up (frequency > 1)
- Replaces all cards which don't have a pair with the next cards in the deck

Once you draw cards, then recompute how many cards match up (what kind of hand it is)

Matlab Code

Start with a 5-card hand

- Lines 17 - 23

Check for a flush

- Lines 37 - 43
- Shows up an NSuit(1) = 5

Check for pairs

- Lines 57..64
- NPairs(1) = max pairs in hand

```
15  for games = 1:1000
16
17      X = rand(1,52);
18      [a,Deck] = sort(X);
19      TopCard = 1;
20      for i=1:5
21          Hand(i) = Deck(TopCard);
22          TopCard = TopCard + 1;
23        end
24
25  for Draw = 0:N_Draw
26
27      Value = mod(Hand,13) + 1;
28      Suit = floor(Hand/13) + 1;
29
30
31      if(Draw < N_Draw)
32
33          % check for flush
34
35          NSuit = 0*Hand;
36
37             for i=1:5
38                for j=1:5
39                   if(Suit(i) == Suit(j))
40                       NSuit(i) = NSuit(i) + 1;
41                   end
42                end
43             end
```

Matlab Code (cont'd)

If you are dealt a flush

- draw no cards
- lines 79-80

If you are dealt 4 of a kind

- draw one card
- lines 81..83

If you are dealt a full house

- draw no cards
- line 84

If you are dealt 3 of a kind

- draw two cards
- lines 85..89

```
79       if(NSuit(1) == 5)
80            disp('Flush');
81       elseif (N(1) == 4) % 4 of a kind
82           Hand(5) = Deck(TopCard);
83           TopCard = TopCard + 1;
84        elseif ((N(1) == 3)*(N(2) == 2))
85        elseif ((N(1) == 3)*(N(2) < 2))
86           Hand(4) = Deck(TopCard);
87           TopCard = TopCard + 1;
88           Hand(5) = Deck(TopCard);
89           TopCard = TopCard + 1;
90        elseif ((N(1) == 2)*(N(1) == 2))
91           Hand(5) = Deck(TopCard);
92           TopCard = TopCard + 1;
93        elseif ((N(1) == 2)*(N(2) < 2))
94           Hand(3) = Deck(TopCard);
95           TopCard = TopCard + 1;
96           Hand(4) = Deck(TopCard);
97           TopCard = TopCard + 1;
98           Hand(5) = Deck(TopCard);
99           TopCard = TopCard + 1;
100       elseif (N(1) < 2)
101          for i=1:5
102             Hand(i) = Deck(TopCard);
103             TopCard = TopCard + 1;
104          end
105       end
106   end
```

Matlab Code (contd)

After N draw steps, compute what type of hand you have

- lines 152..157

```
132      for i=1:5
133         for j=1:5
134             if(Value(i) == Value(j))
135                 Freq(i) = Freq(i) + 1;
136             end
137         end
138      end
139
140      [a,b] = sort(Freq, 'descend');
141      Value = Value(b);
142      Hand = Hand(b);
143      Suit = Suit(b);
144
145      N = zeros(1,13);
146      for n=1:13
147         N(n) = sum(Value == n);
148      end
149
150      [N,a] = sort(N, 'descend');
151
152   if (N(1) == 4) Pair4 = Pair4 + 1; end
153   if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHo
154   if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1
155   if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22
156   if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1
157   if (NSuit(1) == 5) Flush = Flush + 1; end
158
159   end
160
161   disp('4ok, FH, Flush, 3ok, 2pair, pair')
162   [Pair4, FullHouse,Flush,Pair3,Pair22,Pair2]'
163
```

Monte-Carlo Simulation Results

Check the program by comparing the results with 0 draws

- Should match previous calculations
- Not exact but close

	0 Draws 5-Card Stud		1-Draw 5-Card Draw	
	Simulation	Calculation	Simulation	Calculation
4 of a kind	17	24	245	243
full house	152	144	1,190	?
3 of a kind	2,148	2,112	7,824	?
2-pair	4,836	4,753	13,162	?
1-pair	42,102	42,194	51,682	?
High-Card	50,745	50,773	25,897	?

Number of each type of hand in 100,000 games

Poker Variations:

- Add multiple draw rounds
- Implement with a for-loop

	0 Draws 5-card stud	1 Draw 5-Card Draw	2 Draws	3 Draws
4 of a kind	17	245	1,063	2,597
full house	152	1,190	3,329	6,667
3 of a kind	2,148	7,824	13,484	18,043
2-pair	4,836	13,162	21,727	27,687
1-pair	42,102	51,682	47,619	39,211
High-Card	50,745	25,897	12,778	5,795

Number of each type of hand in 100,000 games

Summary

With conditional probabities, you can compute the odds of each hand with 5 card draw

- Uses the results from 5-card stud

You could then use these results to compute the odds with two draw steps

- Uses the results of 5-card draw

Then compute the odds with three draw steps

- Uses the results of d-card draw draw

etc.