Combinations and Permutations

5-Card Stud Poker

ECE 341: Random Processes

Lecture #2

note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Combinations and Permutations: 5-Card Stud

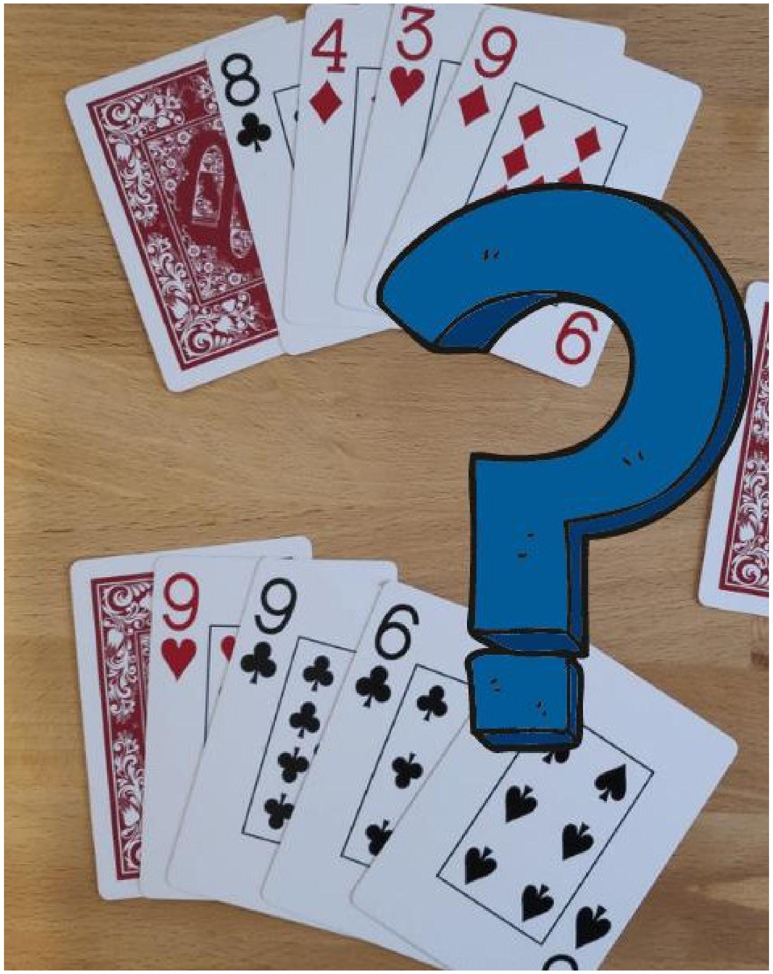
- Combinatorics: how many ways an event can happen.

Definitions:

- n! "n factorial" n x (n-1) x (n-2) x ... x 2 x 1
- 0! = 1 Just define zero factorial to be one.
- p(x) "the probability of outcome x"
- ${}_nP_m$ "Permutations of n events taken m at a time".
  ${}_nP_m = \frac{n!}{(n-m)!}$
- ${}_nC_m$ "Combinations of n events taken m at a time"
  ${}_nC_m = \frac{n!}{m! \cdot (n-m)!}$
- $\binom{n}{m}$ "n choose m". Another way of writing ${}_nC_m$
- Sample With Replacement: Each sample is of the same population size
- Sample Without Replacement: Each time you sample, the remaining population becomes one smaller

Poker: 5-Card Stud

- Starting out, a deck of 52 cards is shuffled.
- The first card is played face down so that only the player sees this card.
- The second card is played face up so all players can see it. After two cards are played, bets are made.
- Once betting stops, a third card is played face up and betting starts over again.
- Ditto for the 4th card and 5th card.
- Once betting is finished with 5 cards for each player, the face down card is revealed and the winner is determined.



Rank of Hands

The winning hands (in order) for poker are

- Royal Flush: 10-J-Q-K-A of the same suit
- Straight-Flush: A run of 5 cards in the same suit
- 4 of a kind: Four of your cards have the same value. Ex: J-J-J-J-x
- Full-House: 3 of a kind and a pair. Ex: J-J-J-Q-Q
- Flush: All cards of the same suit
- Straight: A run of 5 cards
- 3 of a kind: Three of your cards match. Ex: J-J-J-x,y
- 2-Pair: Two pairs of cards. J-J-Q-Q-x
- Pair: Two cards match. J-J-x-y-z
- High-Card: Other. No pairs, no straight, no flush.

Factorials:

How many ways are there to shuffle a deck of 52 cards?

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.
- etc.

The total number of ways a deck can be shuffled is

$$N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot ...2 \cdot 1$$

This is 52 factorial, written as

$$N = 52!$$

Most calculators have a factorial key

$$N = 8.0658 \cdot 10^{67}$$

Note that this is too many combinations for even a computer to run through.

Permutations: (Order Matters)

The number of ways a given hand can play out is..

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.

giving

$$N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$$

Another way to write this is

$$N = {}_{52}P_5 = \left(\frac{52!}{(52-5)!}\right) = 311,875,200$$

There are 311,875,20 different ways a given hand can play out.

Combinations (order doesn't matter)

When determining who won the hand, the order of the cards doesn't matter.

- Select the first card from 5 possibilities
- The second card from 4 possibilities
- The third card from 3 possibilities.
- etc.

or

$$M = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$M = 5! = 120$$

The number of hands in poker is then

$${}_{52}C_5 = \binom{52}{5} = \#hands = \left(\frac{52!}{(52-5)! \cdot 5!}\right) = 2,598,960$$

Probability of getting a Royal Flush

The probability of any given hand as

$$p(x) = \frac{\text{the total number of hands that are x}}{2{,}598{,}960}$$

The easiest is a royal flush:

- There are only four possibilities
  - Spades, hearts, clubs, and diamonds
- $p(\text{royal flush}) = \left(\frac{4}{2{,}598{,}960}\right) = \frac{1}{649{,}740}$

The odds against getting a royal flush are 649,740 : 1

- You should on average get a royal flush once every 649,740 hands.

4 of a kind:

There are several ways to compute this. One ways is as follows:

Assume the cards in your hand are x-x-x-x-y

- There are 13 cards in a deck: choose one
- Of the four cards of that value, choose four
- Of the 48 remaining cards, choose one

$$N = \binom{13}{1}\binom{4}{4}\binom{48}{1} = 624$$

There are 624 different hands that give you a 4-of-a-kind.

- The odds being dealt 4 of a kind is 4165 : 1

$$p(N) = \left(\frac{624}{2{,}598{,}960}\right) = \left(\frac{1}{4165}\right)$$

3-of-a-kind 54,912

- xxx ab
- Of the 13 value, choose one for x
- Of the four x's in the deck, choose three
- Of the 12 remaining values, choose 2 for a and b
- Of the 4 a's, choose one (ditto for b)

$$N = \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$$

The odds against getting three-of-a-kind are 47.33 : 1 against

$$p(N) = \left(\frac{54{,}912}{2{,}598{,}960}\right)$$

p() = 47.33 : 1 odds against

Flush

- Of the four suits, choose one
- Of the 13 cards in that suit, choose five

$$N = \binom{4}{1}\binom{13}{5} = 5,184$$

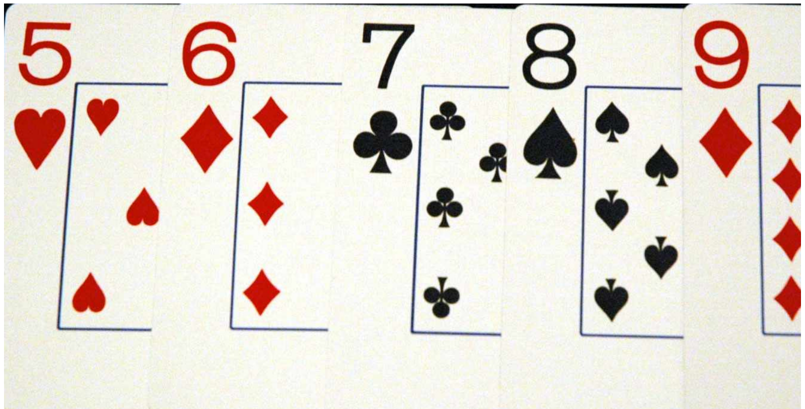
- This includes straight-flushes (N = 40).
  - Removing these gives 5,144 different flushes

The odds against getting a flush are 505.2 : 1

$$p(N) = \left(\frac{5,144}{2,598,960}\right) = \left(\frac{1}{505.2}\right)$$

Straight

- A straight can start with an Ace through a 10.
  - That gives 40 starting cards (10 values in 4 suits)
- Of the four cards of the next value, choose one
- Of the four cards of the next value, choose one



so

$$N = 40 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10,240 \quad \textit{all straights}$$

This also counts straight-flushes

$$N = 40 \cdot \binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1} = 40 \quad \textit{straight flushes}$$

so the total number of straights is

$$N = 10,200$$

giving the probability of 254.8:1 against

2-Pair

- Of the 13 values (Ace through King), pick two
- For the first value, pick two cards of the four in the deck
- For the second value, pick two cards of the four in the deck
- For the last card, pick one (44 choose 1)

$$N = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$$

$$N = 123,552$$

The odds of getting 2-pair are

$$p(x) = \left(\frac{123,552}{2,598,960}\right) = \left(\frac{1}{21.035}\right)$$

or 21.03 : 1 odds against getting two-pair.

Probability of all poker hands

https://allmathconsidered.wordpress.com/2017/05/23/the-probabilities-of-poker-hands/

POKER HAND		COUNT	Odds Against
2	Straight Flush	40	64,974
3	Four of a Kind	624	4,165
4	Full House	3,744	694.17
5	Flush	5,108	508.8
6	Straight	10,200	254.8
7	Three of a Kind	54,912	47.33
8	Two Pair	123,552	21.04
9	One Pair	1,098,240	2.37
10	High Card	1,302,540	2
Total		2,598,960	

Monte-Carlo Simulations

Monte-Carlo simulations are another way to determine probabilities. Here,

- A program is written to generate a random hand of poker.
- The type of hand it is then logged
- This experiment is then repeated a lage number of times (1 million plus)

The odds are then approximately the number of times each hand occurred divided by the number of hands dealt.

Shuffle the deck and draw 5 cards:

```
X = rand(1,52);
[a,Deck] = sort(X);
Deck(1:10)

ans =    47    40    39     3    44    16    17    49     2    25
```

The value of the card is the number mod 13, the suit is the value / 4

```
Hand = Deck(1:5)

Hand =    47    40    39     3    44

Value = mod(Hand-1, 13) + 1

Value =     8     1    13     3     5

Suit = ceil(Hand/13)

Suit =     4     4     3     1     4
```

To determine the number of pairs (2-of-a-kind) etc, count how many times the value of a given card is 1 (Ace), 2, 3, 4, 5, etc

```
N = zeros(1,13);
for n=1:13
    N(n) = sum(Value == n);
end
```

Then determine what kind of hand by

- Sorting the number of matching cards
- Checking if the maximum of the matches is 4 (4-of-a-kind), 3 and 2 (full house), etc

```
[N,a] = sort(N, 'descend');

if (N(1) == 4) Pair4 = Pair4 + 1; end
if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHouse + 1; end
if ((N(1) == 3)*(N(2) < 2))  Pair3 = Pair3 + 1; end
if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; end
if ((N(1) == 2)*(N(2) < 2))  Pair2 = Pair2 + 1; end
```

Matlab Code

- Stud Poker

Play 100,000 hands

- line 11

Shuffle the deck

- lines 13-14

Draw 5 cards

- line 15

Count the frequency of each card

- lines 19-24

Determine what kind of hand it is

- lines 26-30

```
1   % ECE 341 Lecture #2
2   % 5-Card Stud
3   % Combinations and Permutations
4
5   Pair4 = 0;
6   FullHouse = 0;
7   Pair3 = 0;
8   Pair22 = 0;
9   Pair2 = 0;
10
11  for i0 = 1:1e5
12
13  X = rand(1,52);
14  [a,Deck] = sort(X);
15  Hand = Deck(1:5);
16  Value = mod(Hand-1,13) + 1;
17  Suit = ceil(Hand/13);
18
19  N = zeros(1,13);
20  for n=1:13
21      N(n) = sum(Value == n);
22  end
23
24  [N,a] = sort(N, 'descend');
25
26  if (N(1) == 4) Pair4 = Pair4 + 1; end
27  if ((N(1) == 3)*(N(2) == 2)) FullHouse=FullHouse + 1; end
28  if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3 + 1; end
29  if ((N(1) == 2)*(N(2) == 2)) Pair22 = Pair22 + 1; end
30  if ((N(1) == 2)*(N(2) < 2)) Pair2 = Pair2 + 1; end
31
32  end
33
34  disp('        4-of-kind   Full House  3-of-kind   2-Pair
35  disp([Pair4, FullHouse,Pair3,Pair22,Pair2])
```

The results for 1 million hands of poker:

```
          4-of-kind  full-house   3-o-k        2-pair     pair
  X =       252        1409       21097       47498      423055
```

The odds are

Hand	Computed Odds	Simulated Odds
4 of a kind	4165 : 1	3968 : 1
Full-House	694 : 1	709.7 : 1
3 of a kind	47.3 : 1	47.4 : 1
2-pair	21.0 : 1	21.0 : 1
pair	2.37 : 1	2.36 :1

Summary

With combinatorics, you can calculate how many cominations result in a given outcome

- 4 of a kind
- full house
- 3 of a kind

Where there are a large number of possibilities, this cans save a lot of time

- Enumeration can take hours or days to go through every permutation
- Example: There are 635,013,559,600 bridge hands