Conditional Probabilities

Problem: Given a joint PDF

$$f_{X,Y}(x,y) = P(X < x, Y < y)$$

find the PDF for y given information about x:

$$f_{Y|X}(y|x) = P(Y < y|x)$$

(read the probability that Y is less than y given information about x.)

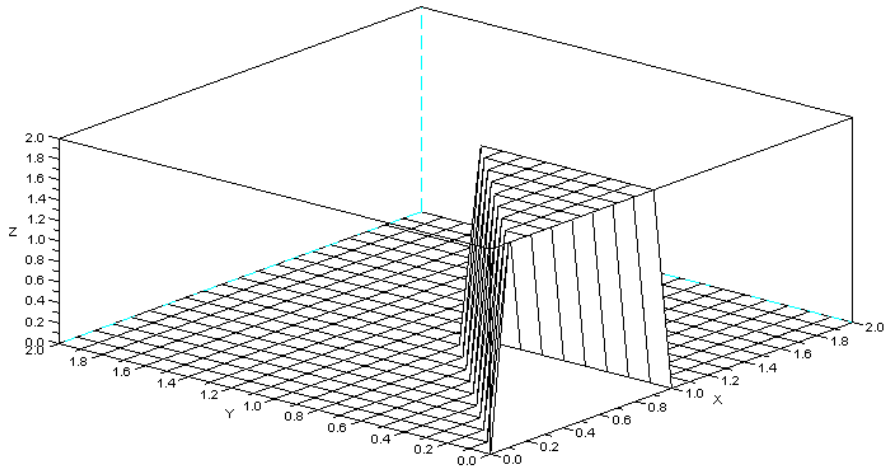
Example: Consider the joint PDF from last lecture:

$$f_{X,Y}(x,y) = 2 \qquad 0 < y < x < 1$$

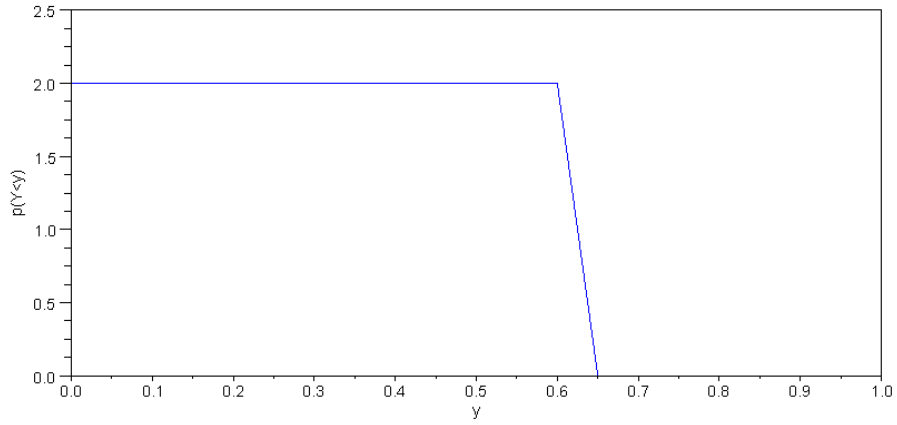
Assume it is known that x=0.6. Find the PDF for y.

$$f_{Y|X}(y|x = 0.6) = P(Y < y|x = 0.6)$$

Pictorially, take the cross section of the PDF along the plane x = 0.6.



This results in the following:



Notes:

- This should be a step function at y = 0.6. The course grid size causes the ramp.
- The area under the curve is 1.2 - this is not a valid PDF.

To make this a valid PDF, you have to scale by a constant. The net result is

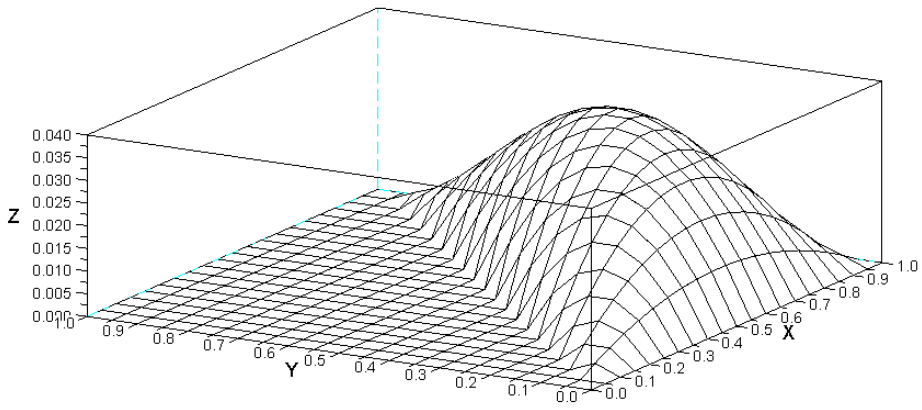
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

or to put it another way

$$f_{Y|X}(y|x=\alpha) = \frac{f_{X,Y}(x=\alpha,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x=\alpha,y)\cdot dy}$$

Example 2: Take the second joint PDF from last lecture:

$$f_{X,Y}(x,y) = \alpha \cdot (1-x) \cdot y \cdot (x-y) \qquad 0 < y < x < 1$$



Find the PDF for $f_Y(y)$ given that x = 0.2.

Again, take the cross section along the line x = 0.2 (or simply plug in x = 0.2). This results in

$$f_{Y|X}(y|x=0.2) = \alpha \cdot (1-0.2) \cdot y \cdot (0.2-y) \qquad 0 < y < 0.2 \ < 1$$

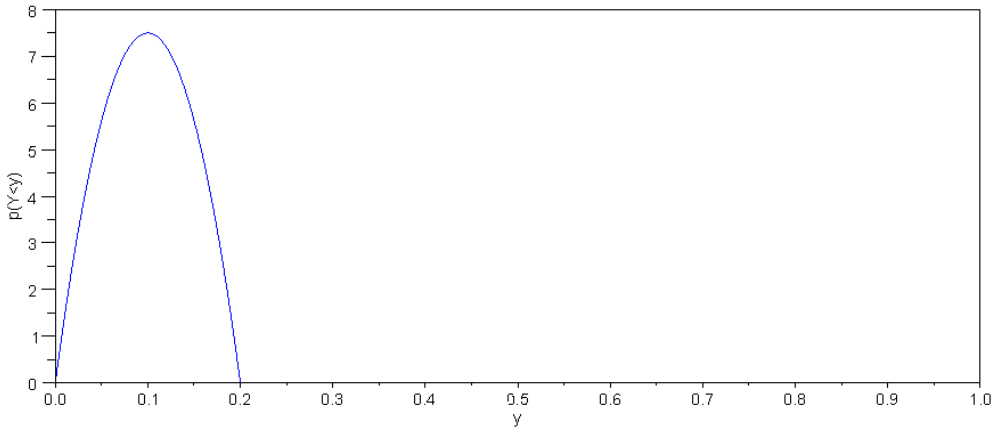
Add in a constant to make the area equal to one (making this a valid PDF)

```
-->y = [0:0.001:1]';

-->fy = y .* (0.2-y);
-->fy = max(0, fy);

-->k = 1 / ( sum(fy) * 0.001)
    750.01875

-->fy = fy * k;
-->plot(y,fy)
```



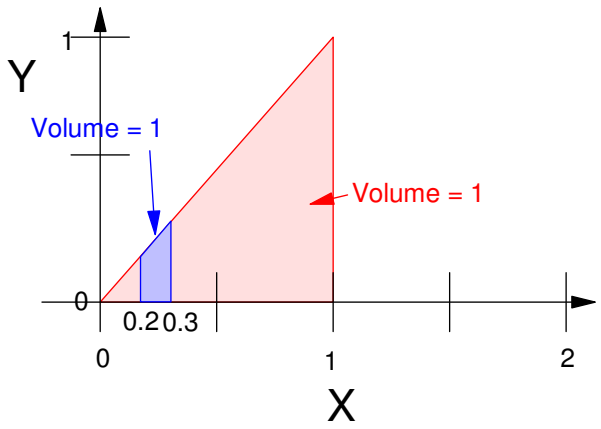
Example 3: Assume

$$f_{X,Y}(x,y) = 2 \qquad 0 < y < x < 1$$

Find the PDF for f(y) given that 0.2 < x < 0.3.

Solution:

- Eliminate the part of the PDF that is out of range. This results the blue region shown below.
- To be a valid PDF, the volume must be one. Scale the resulting PDF until the volume is one.
- For the remaining PDF, integrate over x to find $f_Y(y)$



The area of the trapezoid is (0.1)(0.25) = 0.025. The height must be 40 to be a valid PDF.

Now, integrate over x to find f(y)

$$f_{Y|X}(y|x) = \begin{cases} \int_{x=0.2}^{x=0.3} (40)dv & 0 < y < 0.2 \\ \int_{x=y}^{x=0.3} (40)dv & 0.2 < y < 0.3 \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} (40v)_{v=0.2}^{v=0.3} & 0 < y < 0.2 \\ (40v)_{x=y}^{x=0.3} & 0.2 < y < 0.3 \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} 4 & 0 < y < 0.2 \\ 12 - 40y & 0.2 < y < 0.3 \end{cases}$$