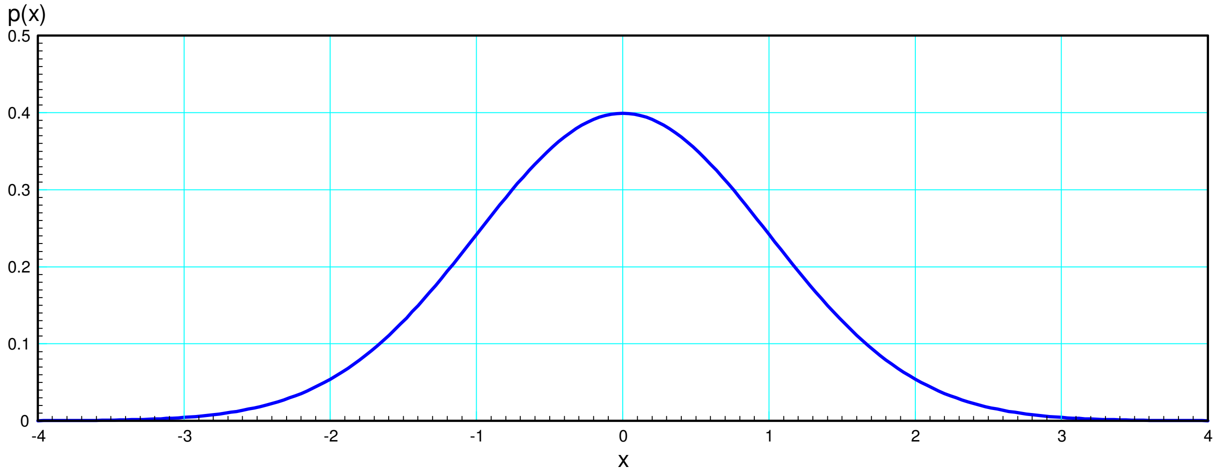
Normal Distribution

a.k.a. Gaussian Distribution



pdf for a normal distribution with a mean of 0, variance of 1

Probably the most important probability distribution is the Normal Distribution. This is the bell-shaped curve you've encountered for class averages, the height or people, IQ scores, etc. The Central Limit Theorem (coming soon) states that

- If you sum random variables together (with certain loose restrictions), the resulting distribution converges to a Normal distribution

Moreover, if you add to random variables together that have a Normal distribution, the result will have a Normal distribution. In short,

- Everything converges to a Normal distribution.
- Once you arrive at a Normal distribution, you're stuck with a Normal distribution.

This is part of the reason Normal distributions are so common.

pdf and mgf:

A Normal distribution is expressed as

$$X \sim N(\mu, \sigma^2)$$

read as

X has a Normal distribution with mean μ and variance σ^2

The pdf for a Normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

while the moment generating function

$$\psi(s) = \exp\left(\mu s + \frac{\sigma^2 s^2}{2}\right)$$

From the moment generating function you can see that

$$m_0 = \psi(0) = 1$$

This is a valid moment generating function

The mean of a Normal distribution is the 1st moment:

$$m_1 = \psi'(0) = \left((\mu + \sigma^2 s)\exp\left(\mu s + \frac{\sigma^2 s^2}{2}\right)\right)_{s=0} = \mu$$

The second moment is

$$m_2 = \psi''(0)$$

$$= \left(\sigma^2 \exp\left(\mu s + \frac{\sigma^2 s^2}{2}\right) + (\mu + \sigma^2 s)^2 \exp\left(\mu s + \frac{\sigma^2 s^2}{2}\right)\right)_{s=0}$$

$$= \sigma^2 + \mu^2$$

The variance is then

$$var = m_2 - m_1^2 = \sigma^2$$

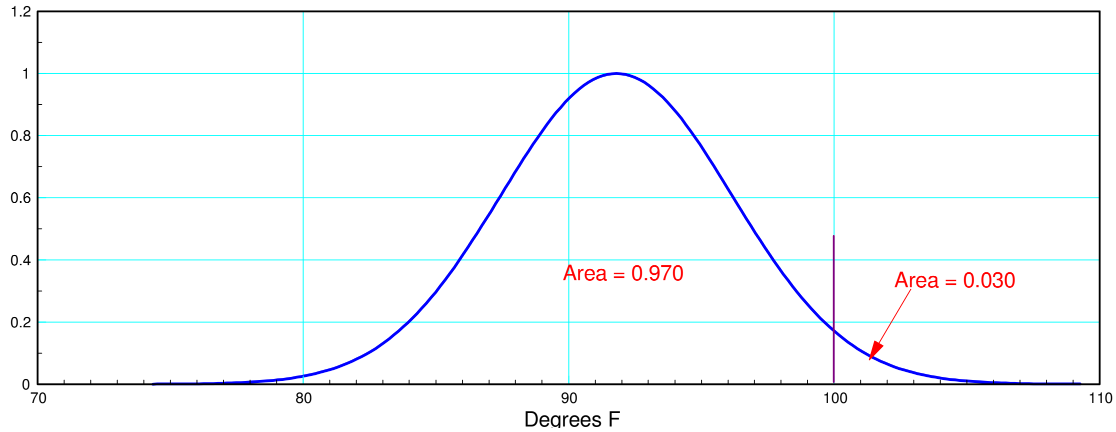
Examples of Normal Distributions

Monthly High Temperature: Hector Airport has been monitoring the temperature in Fargo since 1942. From this data, assume the high for the month of June is Normally distributed[1] with

- mean = 91.7962F
- standard deviation = 4.3657

The pdf for the high for the month of June then looks like:

[1] Note: This is actually a t distribution (coming later) - which is a Normal distribution where you estimate the mean and standard deviation from the data. If you have enough data points, a t distribution converges to a Normal distribution.

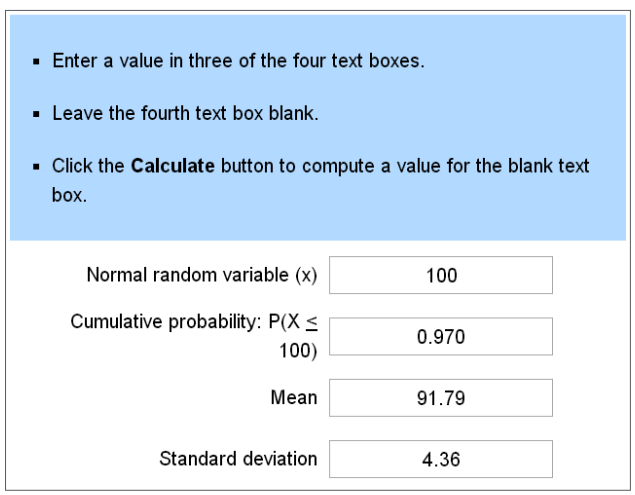


Normalized pdf for the high for the month of June, Fargo ND

From this, you can answer some questions:

What is the probability that the high will be more than 100F this coming year?

This is asking what the area of the curve is to the right of 100F. Here, StatTrek is useful:



https://www.stattrek.com/online-calculator/normal.aspx

The area to the left is 0.970

The area to the right is 0.030

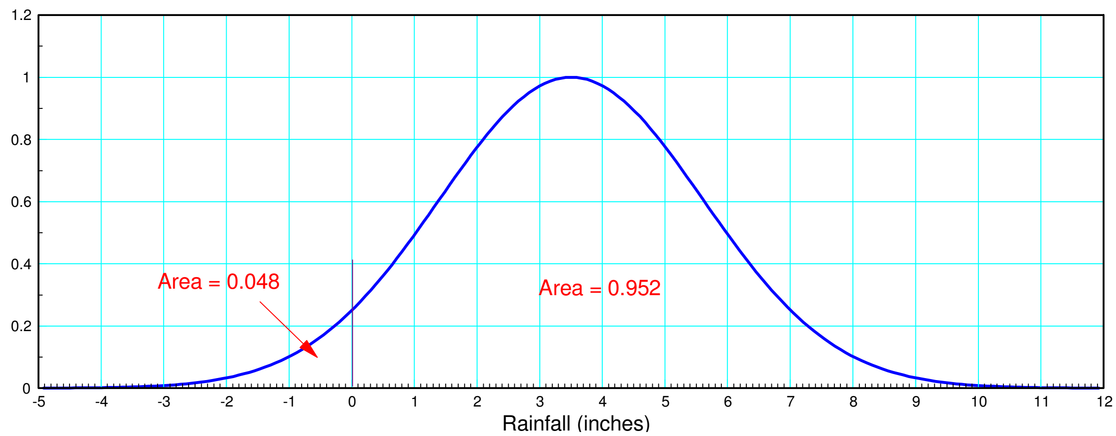
There is a 3.0% chance that it will break 100F this coming June

Rain Fall: The mean and standard deviation for the amount of rainfall in June in Fargo, ND is

- mean = 3.5025 inches
- standard deviation = 2.1054

Assume this is a Normal distribution.

- Plot the pdf

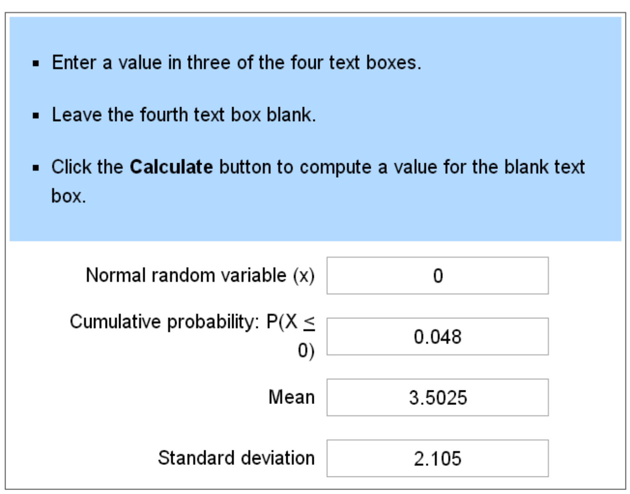


pdf for the rainfall in June

Note that negative rainfall does not make sense. It would actually be more appropriate to use a Poisson distribution for this data.

What is the chance we will receive no rain this coming June?

Sticking with the Normal distribution, the area to the left of 0" is 0.048. There is 4.8% chance we will get no rain this coming June.



https://www.stattrek.com/online-calculator/normal.aspx

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Addition of Normal Distributions:

If you add to Normal distributions, the result is a Normal distribution with

- mean = mean(a) + mean(b)
- variance = variance(a) + variance(b)

Proof: Multiply the moment generating functions

$$\psi_a(s) = \exp\left(\mu_a s + \frac{\sigma_a^2 s^2}{2}\right)$$

$$\psi_b(s) = \exp\left(\mu_b s + \frac{\sigma_b^2 s^2}{2}\right)$$

$$\psi_a(s)\psi_b(s) = \exp\left(\mu_a s + \frac{\sigma_a^2 s^2}{2}\right)\exp\left(\mu_b s + \frac{\sigma_b^2 s^2}{2}\right)$$

$$= \exp\left(\left(\mu_a s + \frac{\sigma_a^2 s^2}{2}\right) + \left(\mu_b s + \frac{\sigma_b^2 s^2}{2}\right)\right)$$

$$= \exp\left((\mu_a + \mu_b)s + \frac{\left(\sigma_a^2+\sigma_b^2\right)s^2}{2}\right)$$

Normal + Normal = Normal

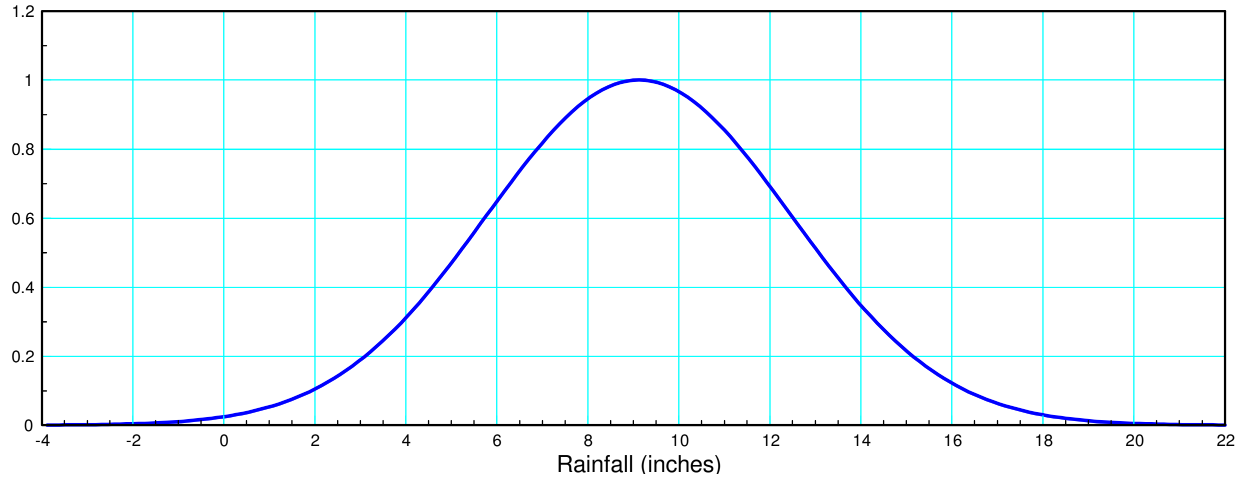
- The means add
- The variance adds

Example: The rainfall for the months of June, July, and August have the following statistics:

Month	Mean	Variance
June	3.5025	4.4327
July	2.9668	3.8044
August	2.6529	3.0063
Sum	9.1221	11.2434

What is the mean and variance for the rainfall for the whole summer (June + July + August)

Solution: Add up the means, add up the variances.



Normalized pdf for the total rainfall over the summer in Fargo

Standard Normal Distribution

The Normal distribution is extremely common and extremely useful. Unfortunately, it is difficult to integrate.

To get around this, tables showing the area under the curve as you move away from the mean could be used. Unfortunately, there are an infinite number of possible means and standard deviations.

To get around this, the Standard Normal Distribution is used. This is a Normal distribution with

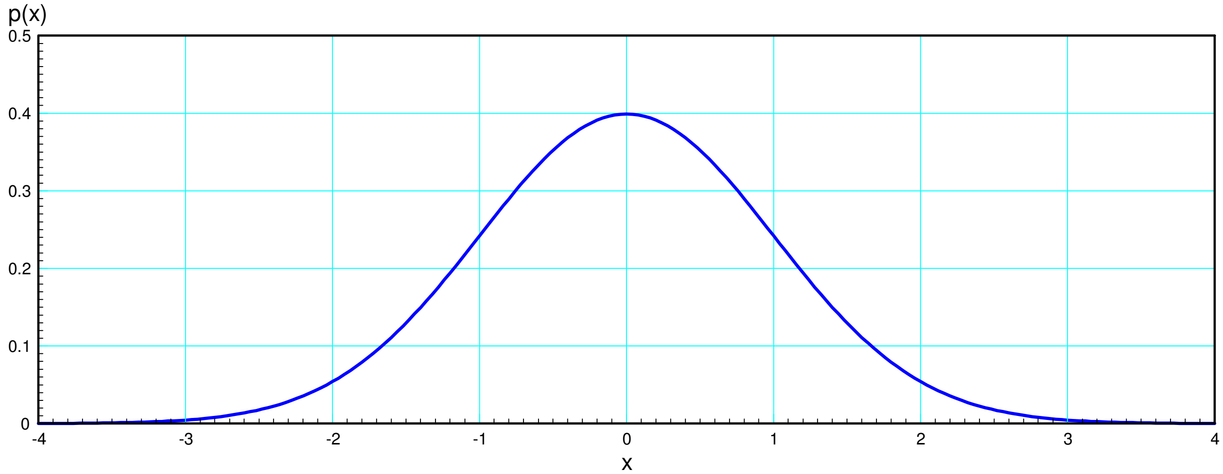
- mean = 0
- standard deviation = 1

With this stipulation, you can come up with a table showing the area under the curve as you move away from the mean. Once known, you can easily convert this to the distribution you care about.

The standard normal distribution has the following pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)}$$

It's shape is a nice bell curve:



pdf for a Normal distribution with mean = 0, standard deviation = 1 (a.k.a. a Standard Normal Distribution)

The area under the tail as you move away from the mean is then

z	0	-1	-2	-3	-4	-5
p(x<z)	0.5	0.1587	0.0227	0.001349	3.167 10-5	2.866 10-7

z	-1.28	-1.64	-1.96	-2.24	-2.33	-2.58
p(x<z)	0.1	0.05	0.025	0.0125	0.01	0.005

The z-score is how far your value is from the mean in terms of standard deviations

$$z = \left(\frac{x-\mu}{\sigma}\right)$$

This converts your data (x) into a standard Normal distribution (z).

Example: Recall that the high for the month of June has

- mean = 91.7962F
- standard deviation = 4.3657

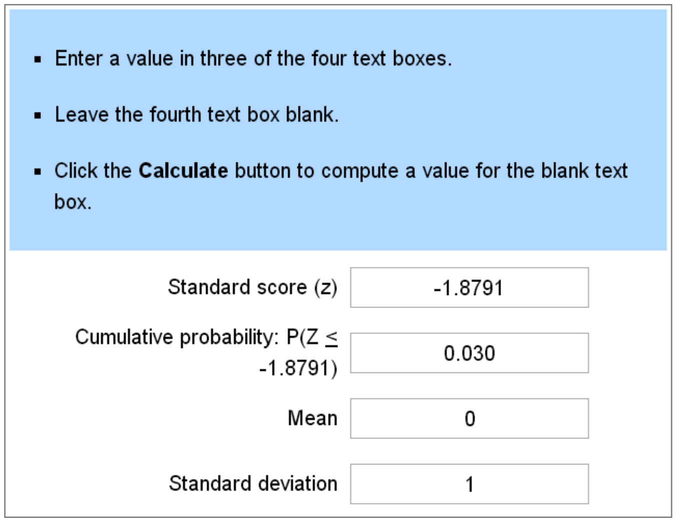
Determine the probability that it will break 100F this coming June.

Solution: Determine the z-score corresponding to 100F

$$z = \left(\frac{100F - 91.7962}{4.3658}\right) = 1.8791$$

From the table at the end, 1.8751 has an area (p(x<z)) of 0.0301

You can also use StatTrek



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The area is 0.030 or 3.0% (same answer as before)

Standard Normal Table: z-score

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.9	0.0001	0.0001	0	0	0	0	0	0	0	0
-3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.6	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.001	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3	0.0014	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.001	0.001
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.002	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.003	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0042	0.004	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.006	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.008	0.0078	0.0076	0.0073	0.0071	0.007	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0126	0.0122	0.0119	0.0116	0.0113	0.011
-2.1	0.0179	0.0174	0.017	0.0166	0.0162	0.0158	0.0154	0.015	0.0146	0.0143
-2	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.025	0.0244	0.0239	0.0233
-1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.063	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1094	0.1075	0.1057	0.1038	0.102	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.123	0.121	0.119	0.117
-1	0.1587	0.1563	0.1539	0.15	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.166	0.1635	0.1611
-0.8	0.2119	0.209	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.242	0.2389	0.2358	0.2327	0.2297	0.2266	0.2236	0.2207	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2644	0.2611	0.2579	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.305	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.281	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.33	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.352	0.3483
-0.2	0.4207	0.4168	0.4129	0.4091	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0	0.5	0.496	0.492	0.488	0.4841	0.4801	0.4761	0.4721	0.4681	0.4641

https://www.math.arizona.edu/~rsims/ma464/standardnormaltable.pdf