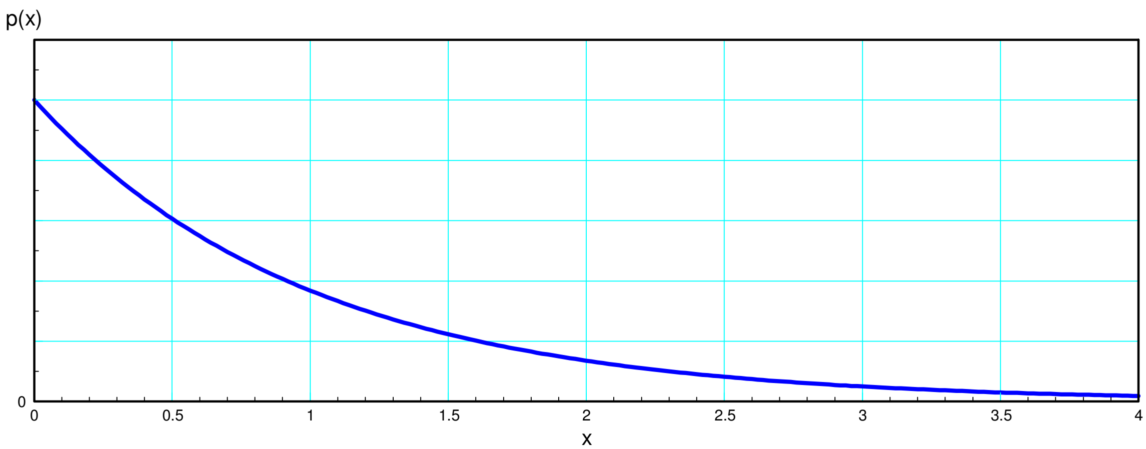
Exponential Distribution

Another common continuous probabilty distribution is the exponential distribution. With this distribution, the pdf is exponential:

$$f_X(x) = \begin{cases} a\,e^{-ax} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$



pdf for an exponential distribution

Examples of where this type of distibution is encountered is:

- Probability that the length of a telephone call is less than x minutes
- Probability that the next atom will decay within x seconds
- Time it takes for the next customer to arrive at a store

Exponential distributions also lead into queueing theory: how long a customer will have to wait in line to be served.

Parameters of Exponential Distributions

pdf: As stated before

$$pdf(x) = a\,e^{-ax}$$

cdf: The cdf is the integral of the pdf

$$cdf(x) = \int_0^x (a\,e^{-at})\,dt$$

$$cdf(x) = 1 - e^{-ax} \qquad x > 0$$

Moment Generating Function: Start with this since it makes everything else a lot easier

$$\psi(s) = \left(\frac{a}{s+a}\right)$$

To be a valid probability, the area under the curve (i.e. the zeroth moment) must be 1.00000

$$\psi(s=0) = 1 = \left(\frac{a}{a}\right)$$

Mean: The mean of an exponential distribution is

$$\bar{x} = \int_0^\infty p(x)\, x\, dx$$

$$\bar{x} = \int_0^\infty (ae^{-ax})\, x\, dx$$

$$\bar{x} = \left(\frac{1}{a}e^{-ax}(-ax-1)\right)_0^\infty$$

$$\bar{x} = \left(\frac{1}{a}\right)$$

This can also be found using the moment generating funciton

$$\psi(s) = \left(\frac{a}{s+a}\right)$$

$$\bar{x} = m_1 = \psi'(0)$$

$$\bar{x} = \left(\frac{a}{(s+a)^2}\right)_{s=0} = \frac{1}{a}$$

Variance: Use the moment generating function - it's a lot easier

$$s^2 = m_2 - m_1^2$$

$$m_2 = \psi''(0)$$

$$m_2 = \left(\frac{2a(s+a)}{(s+a)^4}\right)_{s=0} = \left(\frac{2}{a^2}\right)$$

so

$$s^2 = \left(\frac{2}{a^2}\right) - \left(\frac{1}{a}\right)^2$$

$$s^2 = \left(\frac{1}{a^2}\right)$$

Matlab Example:

Generate 10 random variables with an exponential distibution where a = 1

Solution: Determine the probability using the *rand* function in matlab

```
p = rand
```

Convert to x using the cdf

$$p = cdf(x) = 1 - e^{-ax}$$

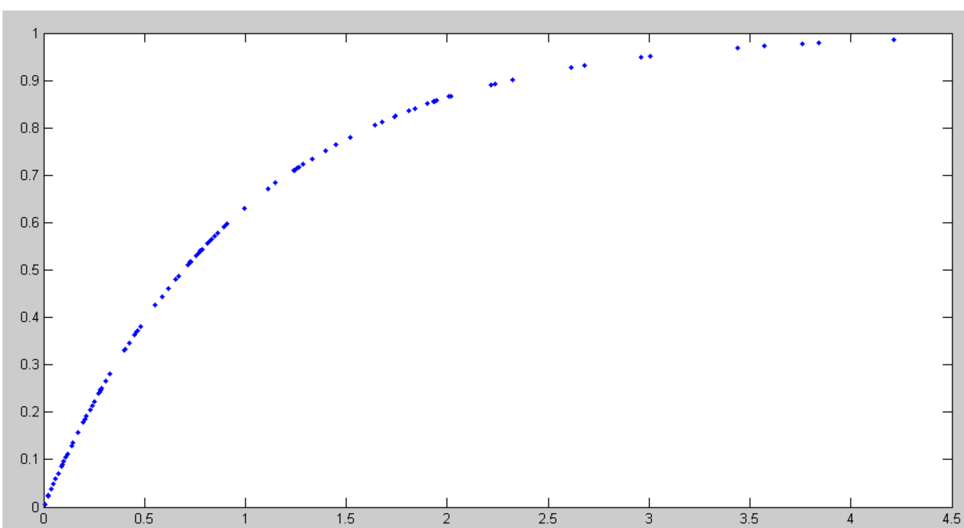
$$x = -\left(\frac{1}{a}\right)\ln(1-p)$$

In matlab

```
p = rand(10,1);

a = 1;
x = -(1/a) * log(1-p)

[p, x]

    0.6862    1.1590
    0.9593    3.2004
    0.4850    0.6635
    0.7880    1.5511
    0.7550    1.4066
    0.7228    1.2831
    0.2646    0.3074
    0.6884    1.1660
    0.9323    2.6930
    0.0902    0.0946
```

Plotting these together give you the cdf: (plotted with 100 points for x)

```
plot(p,x,'.')
```



Plotting x vs. probability gives the cdf for an exponential distribution

Example: A block of radioactive material is sitting next to a Geiger counter. In 30 minutes, the Geiger counter detects 100 atoms decaying. Determine the pdf and CDF for the time until the next atom decays (in minutes).

Solution: The average number of atoms decaying per minute is

$$\bar{x} = \frac{100 \text{ atoms}}{30 \text{ minutes}} = 3.33 \frac{\text{atoms}}{\text{min}} = \frac{1}{a}$$

$$a = 0.3$$

pdf:

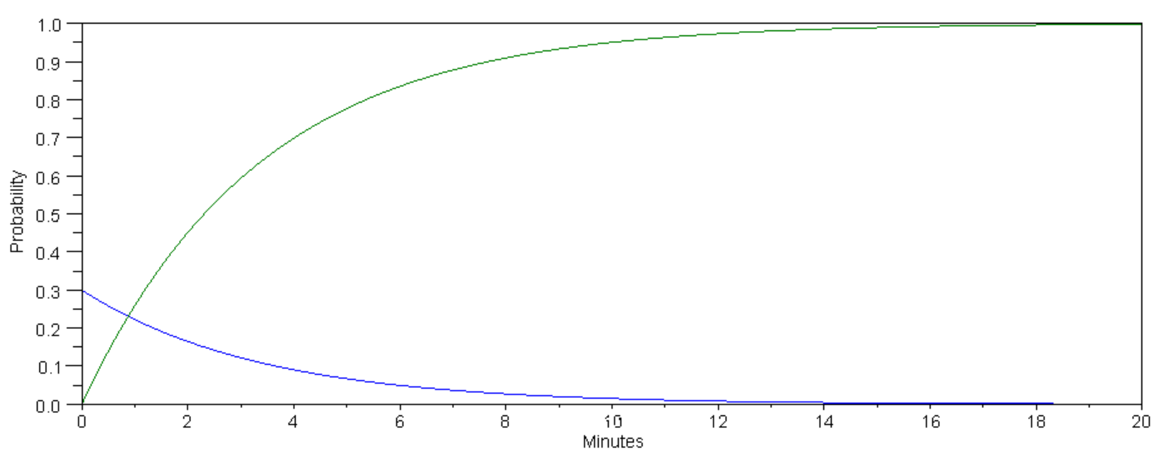
$$f_X(x) = \begin{cases} 0.3 \cdot e^{-0.3x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

CDF:

$$F_X(x) = \begin{cases} 1 - e^{-0.3x} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

Matlab

```
x = [0:0.01:20]';

p = 0.3 * exp(-0.3*x);

C = 1 - exp(-0.3*x);

plot(x,p,x,C)
xlabel('Minutes');
ylabel('Probability');
```



pdf (blue) and CDF (green) for time until you detect an atom decaying

Queing Theory

One place where exponential distributions are often used are in determining how many servers are needed at a restaraunt.

For example, assume

- Customers arrive at a fast-food restaraunt with
  - An exponential distribution, with
  - An average of one customer every minute.
- It takes an average of 30 seconds to serve each customer.

How long will the longest wait be for a 1 hour shift?

Start with generating 1 hours worth of custumers:

```
a = 1/60;
s = 0;
x = [];
Tarr = []
n = 0;
t = 0;

while(t < 3600)
    n = n + 1;
    p = rand;
    x = -(1/a)*log(1-p);
    t = t + x;
    TIME(n) = t;
    disp([n, p, x, t])
end
```

58 customers arrived over this hour with arrival times.

As each customer arrives, their service time is the next available slot:

- 30 seconds after the last custimer is served, or
- Immediately if there is no wait.

The wait time is the difference in time from being served to time of arrival

The max queue size is the number of customers waiting to be served at the time you arrive (their finish time is more than your arrival time)

What you want is

- To have the fewest servers needed, while
- Keeping the wait time less than some threshold (such as 2 minutes. If a customer has to wait more than X about of time, they'll leave)
- Keeping the queue size less than some threshold (if too many people are in line, customers will leave).

Customer	Arrival Time	Serve Time	Finish Time	Wait Time	Queue Size
1	33	33	63	0	0
2	63	63	93	0	0
3	70	93	123	23	1
4	89	123	153	34	2
5	118	153	183	35	2
6	421	421	451	0	0
7	454	454	484	0	0
8	473	484	514	11	1
9	476	514	544	38	2
10	476	544	574	68	3
11	477	574	604	97	4
12	484	604	634	120	4
13	544	634	664	90	3
14	547	664	694	117	4
15	692	694	724	2	1
16	722	724	754	2	1
17	743	754	784	11	1
18	785	785	815	0	0
19	819	819	849	0	0
20	823	849	879	26	1
21	957	957	987	0	0
22	1,240	1,240	1,270	0	0
23	1,249	1,270	1,300	21	1
24	1,322	1,322	1,352	0	0
25	1,391	1,391	1,421	0	0
26	1,552	1,552	1,582	0	0
27	1,572	1,582	1,612	10	1
28	1,608	1,612	1,642	4	1
29	1,734	1,734	1,764	0	0
30	1,747	1,764	1,794	17	1
31	1,798	1,798	1,828	0	0
32	1,829	1,829	1,859	0	0
33	1,861	1,861	1,891	0	0
34	1,873	1,891	1,921	18	1
35	1,945	1,945	1,975	0	0

Erlang Distribution:

$$f_X(x) = \begin{cases} \frac{a^n x^{n-1} e^{-ax}}{(n-1)!} & 0 < x < \infty \\ 0 & otherwise \end{cases}$$

Examples: (CDF)

- Probability that the total time of n telephones calls are less than x
- Probability that n atoms will decay within x seconds

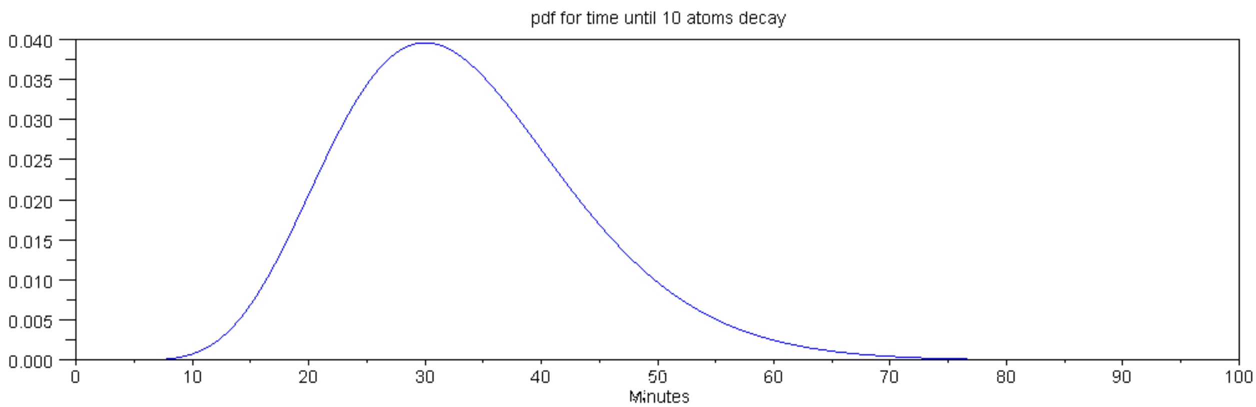
Mean: $E(x) = \frac{n}{a}$

Variance: $\sigma^2 = \frac{n}{a^2}$

Problem: Plot the pdf and cdf for the the time you have to wait until 10 atoms decay.

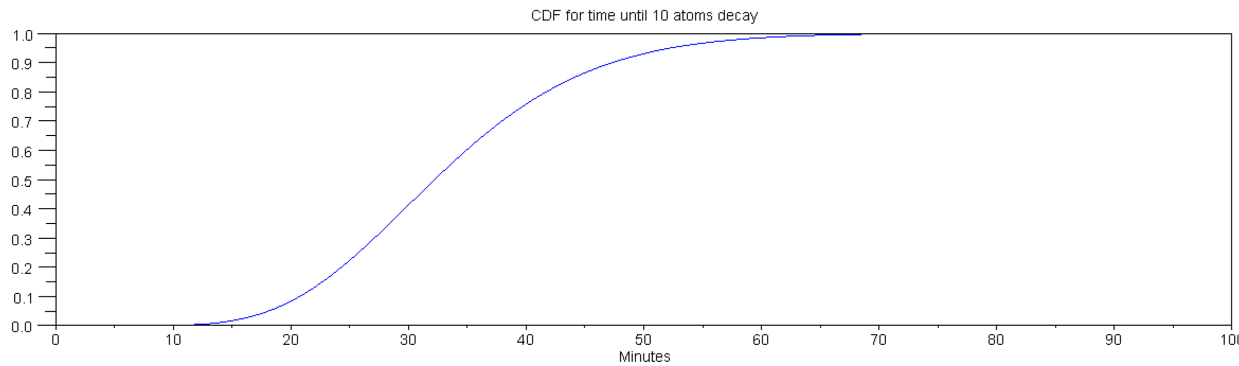
In SciLab, the pdf is found from:

```
-->n = 10;
-->a = 0.3;
-->x = [0:0.01:100]';

-->f = (a^n)*(x .^ (n-1)) .* (exp(-a*x)) / factorial(n-1);

-->plot(x,f)
```



The CDF can be found by integrating (the step size used was 0.01 minute):

```
-->F = 0*f;

-->for i=2:length(f)
-->   F(i) = F(i-1) + f(i)*0.01;
-->   end

-->plot(x,F)
```


CDF for time until 10 atoms decay
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
10
20
30
40
50
60
70
80
90
100
Minutes