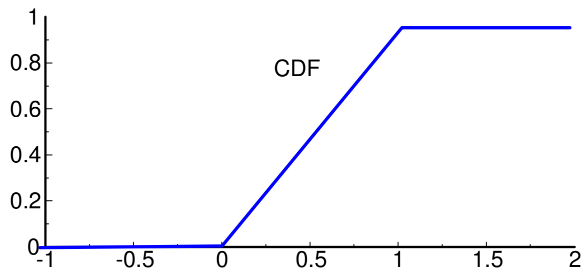
Uniform Distribution

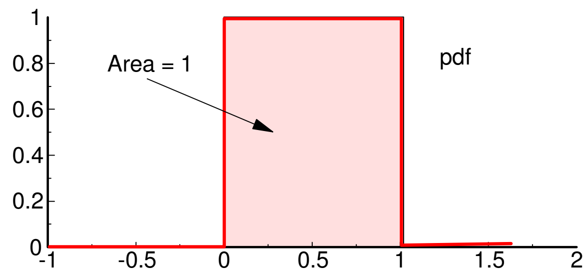
Introduction:

A uniform distribution is one where the probability density function (pdf) is

- constant over a range (a, b), and
- zero otherwise.

For example, a variable X which has a uniform distribution function over the interval of (1, 2) is as follows:





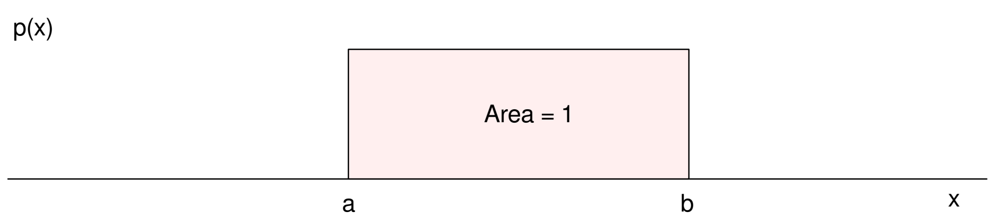
There are several situations where you encounter uniform distributions:

- A 1k resistor with a tolerance of 5% will have a value in the range of +/- 5%. This can be modeled as a uniform distribution over the range of 950 < R < 1050
- A 3904 transistor has a nominal gain of 200 with a variation of +/- 100. This can be modeled as a unifiom distrubtion over the range if 100 < gain < 30

In general, if you know the limits but know nothing else about the distribution, a uniform distribution is a reasonable assumption to make about the random variable.

Properties of Uniform Distributions:

Assume a uniform distribution over the interval (a, b)



Since the area must be 1.000, the height must be

Area = width * height = 1

$$p(x) = \left(\frac{1}{b-a}\right) \qquad a < x < b$$

Mean: The mean of the funciton (almost by inspection) is

$$\mu = \left(\frac{a+b}{2}\right)$$

Variance: The mean does not affect the variance. If you assume a = -b, then

$$\sigma^2 = \int_{-b}^{b} p(x)\,(x-\bar{x})^2\,dx$$

$$= \int_{-b}^{b} \left(\frac{1}{2b}\right)\,(x)^2\,dx$$

$$= \left(\frac{1}{6b}\right)\,(x^3)_{-b}^{b}$$

$$= \left(\frac{b^2}{3}\right)$$

b is 1/2 of the width (b-a) in ths case, so in general the variance is 1/12th of the width

$$\sigma^2 = \left(\frac{(b-a)^2}{12}\right)$$

Moment Generating Function

$$\psi(s) = \left(\frac{1}{s}\right)(e^{-bs} - e^{-as})$$

Combinations of Uniform Distributions

If you add two uniform distributions, the result is

- The colvolution of their pdf's, or
- The producto of their moment generating functions.

Example: Assume A and B are uniform distributions over the interval (0, 1). Find the pdf of the sum Y = A + B.

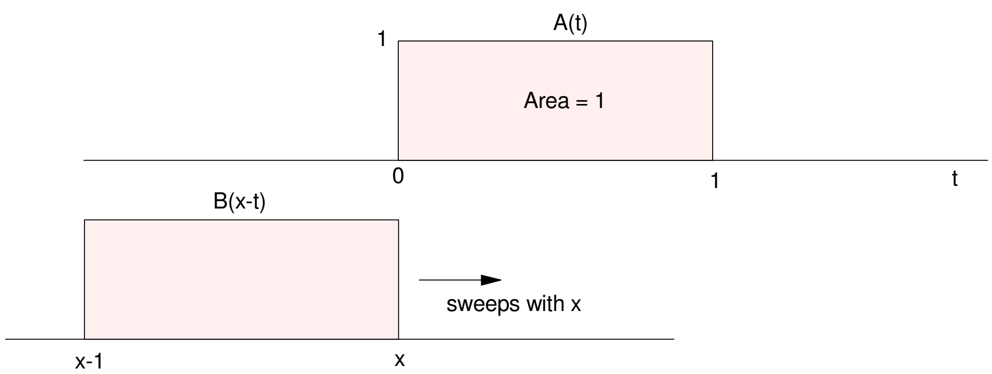
Solution using convolution:

$$A(x) = u(x) - u(x-1)$$

$$B(x) = u(x) - u(x-1)$$

$$Y(x) = A(x) ** B(x)$$

$$Y(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt$$



From the graph, this integral is

Y(x) =

- 0 x < 0
- x 0 < x < 1
- 1 - x 1 < x < 2
- 0 2 < x

This can be done with the functions as well

$$Y(x) = \int_{-\infty}^{\infty} A(t)\ B(x-t)\ dt$$

$$Y(x) = \int_{-\infty}^{\infty} (u(t-1) - u(t))\ (u(x-t-1) - u(x-t))\ dt$$

Note that B(x - t) is zero for

x - t < 0	t > x
x - t > 1	t < x-1

This allows us to simplify the integral

is one in the range of (0, 1), zero otherwise. This allows us to simplify:

$$Y(x) = \int_{x-1}^{x} (u(t) - u(t-1))\, dt$$

Integrating each part

$$Y(x) = (t\, u(t))_{x-1}^{x} - ((t-1)u(t-1))_{x-1}^{x}$$

solving should give your the same answer...

Solve using moment generating functions

$$A(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

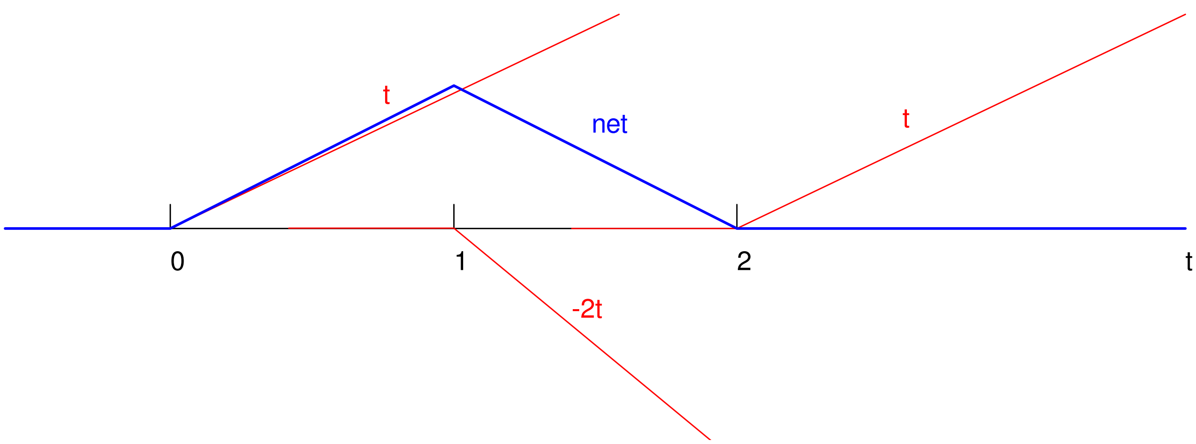
$$B(s) = \left(\frac{1}{s}\right)(1 - e^{-s})$$

$$Y(s) = A(s)B(s)$$

$$Y(s) = \left(\frac{1}{s^2}\right)(e^{-2s} - 2e^{-s} + 1)$$

Take the inverse LaPlace transform

$$y(x) = (x-2)\, u(x-2) - 2(x-1)\, u(x-1) + x\, u(x)$$



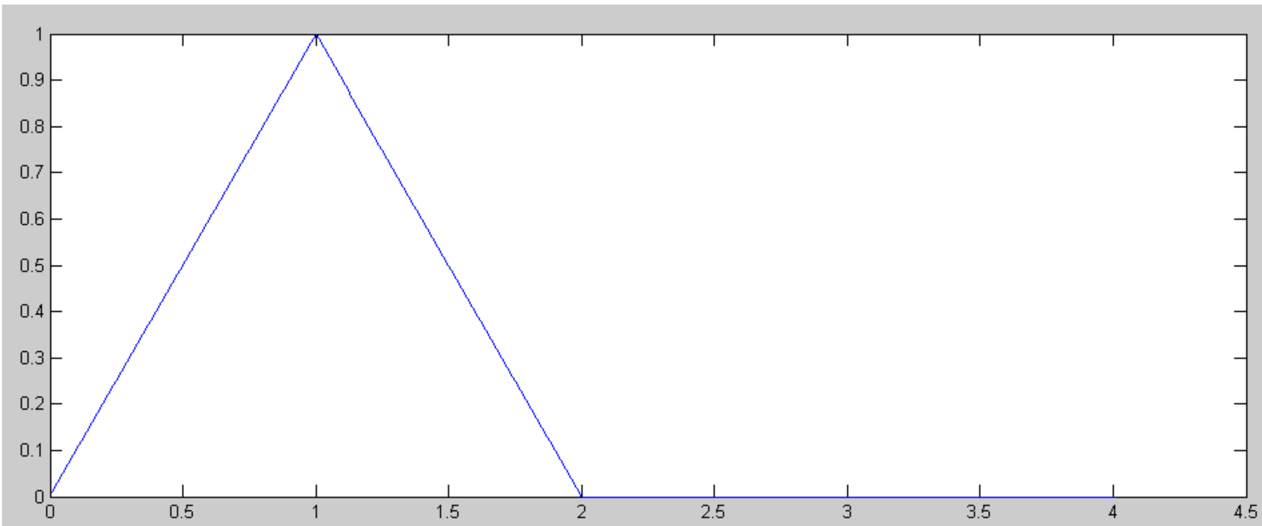
Solve using Matlab: Approximate a uniform distribution with 100 points over the interval (0, 1)

```
dx = 0.01;
x = [0:dx:2]';;

A = 1*(x<1);
B = 1*(x<1);

Y = conv(A, B) * dx;

plot([1:length(Y)]*dx,Y)
```



pdf for the sum of two uniform distributions

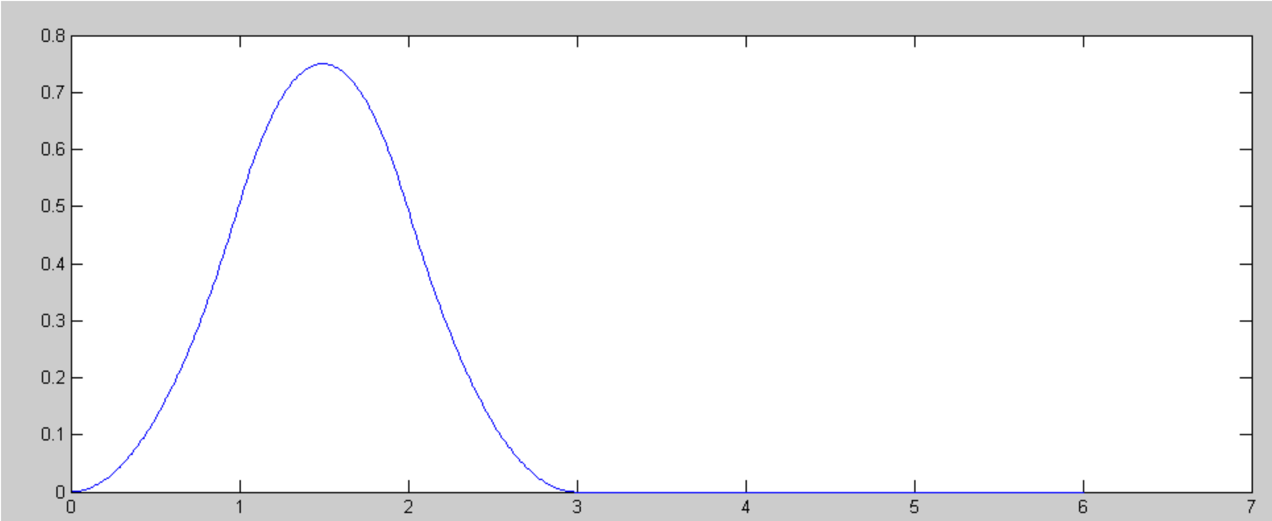
Example 2: Find the pdf of summing three uniform distribitions

- A = uniform(0, 1)
- B = uniform(0, 1)
- C = uniform(0, 1)
- Y = A + B + C

In matlab:

```
x = [0:dx:2]';
A = 1*(x<1);
B = 1*(x<1);
C = 1*(x<1);

Y = conv(A, B) * dx;
Y = conv(Y, C) * dx;
plot([1:length(Y)]*dx,Y)
```

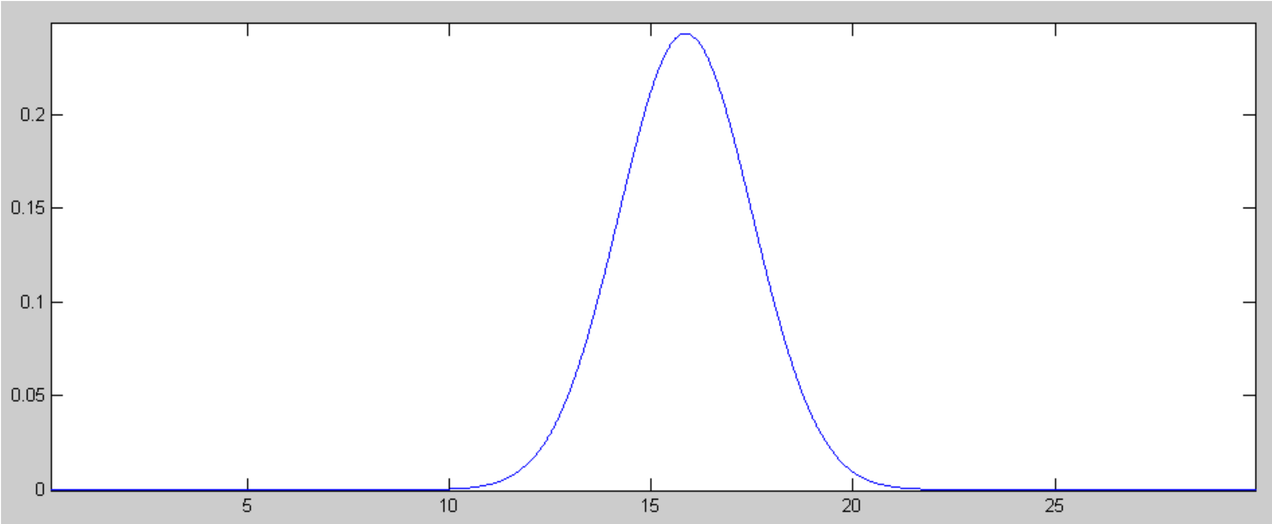


pdf for the sum of three uniform distributions

Example 3: Find the pdf for the sum of 32 uniform distributions:

```
x = [0:dx:2]';
A = 1*(x<1);

Y2 = conv(A, A) * dx;
Y4 = conv(Y2, Y2) * dx;
Y8 = conv(Y4, Y4) * dx;
Y16 = conv(Y8, Y8) * dx;
Y32 = conv(Y16, Y16) * dx;
plot([1:length(Y32)]*dx,Y32)
```



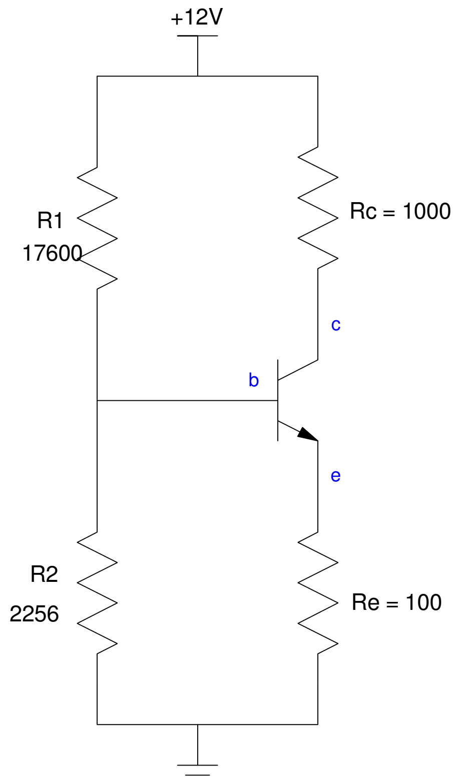
pdf for the sum of 32 uniform distributions

Note that the central limit theorem is evident here: the distribtion is approaching a normal distribution.

Uniform Distribution in Circuit Analysis:

Determine the pdf for Vce. Assume

- Resistors have a 5% tolerance
- The transistor has a gain in the range of 100 to 300



The equations for this circuit (from ECE 321)

$$V_{th} = \left(\frac{R_2}{R_2+R_1}\right)12V$$

$$R_{th} = \left(\frac{R_1 R_2}{R_1+R_2}\right)$$

$$I_b = \left(\frac{V_{th}-.07}{R_{th}+(1+\beta)R_e}\right)$$

$$I_c = \beta I_b$$

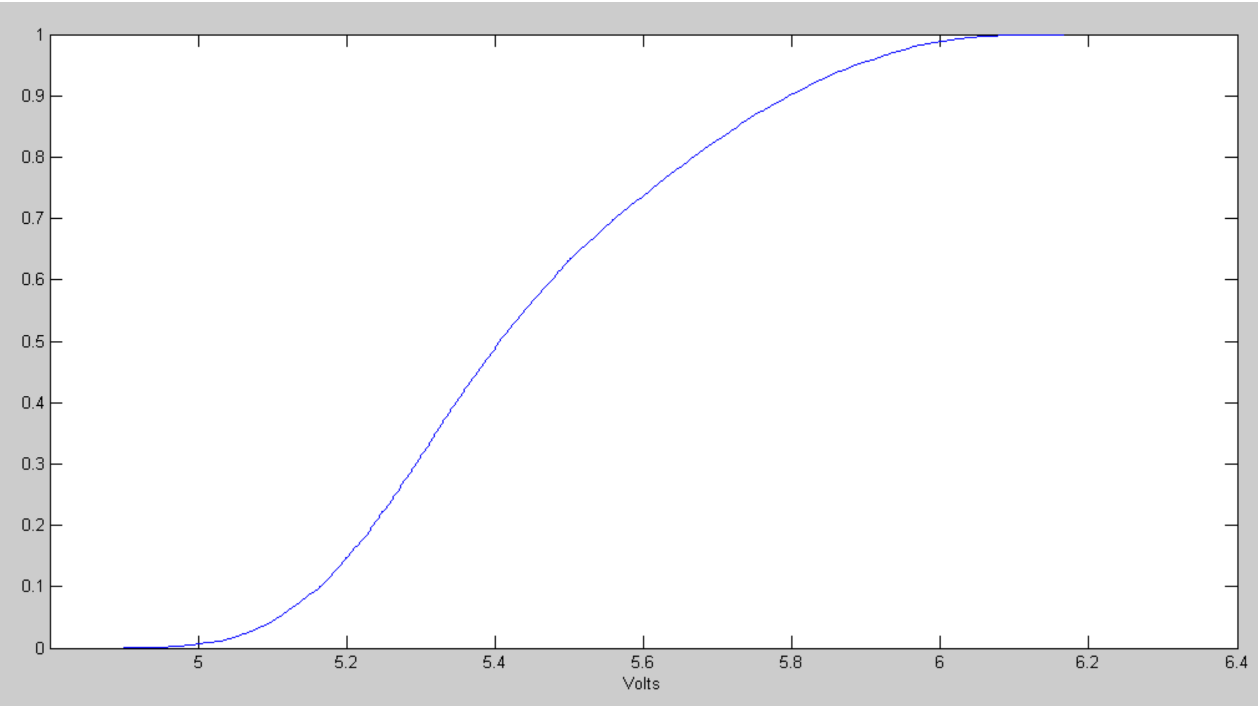
$$V_{ce} = 12 - R_c I_c - R_e(I_b + I_c)$$

Using a Monte-Carlo simulation

```
DATA = [];

for i=1:1000
 R1 = 17600 * (1 + (rand()*2-1)*0.01);
 R2 = 2256 * (1 + (rand()*2-1)*0.01);
 Rc = 1000 * (1 + (rand()*2-1)*0.01);
 Re = 100 * (1 + (rand()*2-1)*0.01);
 Beta = 200 + 100*(rand()*2-1);
 Vb = 12*(R2 / (R1+R2));
 Rb = 1/(1/R1 + 1/R2);
 Ib = (Vb-0.7) / (Rb + (1+Beta)*Re);
 Ic = Beta*Ib;
 Vce = 12 - Rc*Ic - Re*(Ic+Ib);
 DATA = [DATA; Vce];
end


p = [1:length(DATA)]' / length(DATA);
plot(DATA, p)
```



Resulting distribution of Vce from a Monte Carlo simulation

Note that when you sort the data, the resulting plot is the cdf. The pdf is the derivative of this graph