

LaPlace Transforms

Table of LaPlace Transforms

Source: CRC Handbook of Mathematical Tables, CRC Press, 1964

Common LaPlace Transforms		
Name	Time: $y(t)$ ($t > 0$)	LaPlace: $Y(s)$
delta (impulse)	$\delta(t)$	1
unit step	1	$\left(\frac{1}{s}\right)$
unit ramp	t	$\left(\frac{1}{s^2}\right)$
unit parabola	t^2	$\left(\frac{2}{s^3}\right)$
	t^n	$\left(\frac{n!}{s^{n+1}}\right)$
decaying exponential	e^{-bt}	$\left(\frac{1}{s+b}\right)$
	$t e^{-bt}$	$\left(\frac{1}{(s+b)^2}\right)$
	$t^2 e^{-bt}$	$\left(\frac{2}{(s+b)^3}\right)$
	$t^n e^{-bt}$	$\left(\frac{(n-1)!}{(s+b)^{n+1}}\right)$
	$2a \cdot e^{-bt} \cos(ct - \theta)u(t)$	$\left(\frac{a \angle \theta}{s+b+jc}\right) + \left(\frac{a \angle -\theta}{s+b-jc}\right)$

Transfer Functions and Differential Equations:

LaPlace transforms assume all functions are in the form of

$$y(t) = \begin{cases} a \cdot e^{st} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

This results in the derivative of y being:

$$\frac{dy}{dt} = s \cdot y(t)$$

This lets you convert differential equations into transfer functions and back.

Example 1: Find the transfer function that corresponds to the following differential equation:

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Solution: Substitute 's' for $\frac{d}{dt}$

$$s^3 Y + 6s^2 Y + 11sY + 6Y = 8sX + 10X$$

Solve for Y

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

The transfer function from X to Y is

$$G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)$$

Note: The transfer function is often called 'G(s)' since it is the gain from X to Y.

Example 2: Given the transfer function G(s),

$$G(s) = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)$$

determine the differential equation relating X and Y.

Solution: Go backwards. X and Y are related by:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

Cross multiply:

$$(s^3 + 6s^2 + 11s + 6)Y = (8s + 10)X$$

Note that 'sY' means 'the derivative of Y'

$$\frac{d^3 y}{dt^3} + 6\frac{d^2 y}{dt^2} + 11\frac{dy}{dt} + 6y = 8\frac{dx}{dt} + 10x$$

Sidelight: Fractional powers are not allowed in transfer functions.

- $s^2 Y$ means 'the second derivative of Y'.
- $s^{2.3} Y$ means 'the 2.3th derivative of Y'.

I have no idea what a 0.3 derivative is.

Solving Transfer Functions with Sinusoidal Inputs

Example 3: Find $y(t)$ given

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right) X$$

and

$$x(t) = 3 \cos(4t) + 5 \sin(4t)$$

Solution: This is actually a phasor problem. From Euler's identity

$$e^{j4t} = \cos(4t) + j \sin(4t)$$

Take the real part and you get cosine. If you multiply by a complex number

$$(a + jb) e^{j4t} = a \cos(4t) - b \sin(4t) + j(\text{stuff})$$

Take the real part and you get the phasor representation of a sine wave

$$a + jb \leftrightarrow a \cos(4t) - b \sin(4t)$$

Going back to the original problem

- $G(s)$ is valid for all s
- $X(s)$ only exists at $s = j4$, so

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j4} (3 - j5)$$

$$Y = (-0.181 - j0.315)(3 - j5)$$

$$Y = -2.120 - 0.040$$

This is the phasor representation for $y(t)$

$$y(t) = -2.120 \cos(4t) + 0.040 \sin(4t)$$

Example 4: Find $y(t)$ if

$$x(t) = 6 \cos(10t)$$

Same as before but $s = j20$ now:

$$Y = \left(\frac{8s+10}{s^3+6s^2+11s+6} \right)_{s=j10} (6 + j0)$$

$$Y = (-0.067 - j0.034)(6 + j0)$$

$$Y = -0.404 - j0.202$$

which is the phasor representation for

$$y(t) = -0.404 \cos(10t) + 0.202 \sin(10t)$$

Example 5: Find $y(t)$ if

$$x(t) = 3 \cos(4t) + 5 \sin(4t) + 6 \cos(10t)$$

Solution: Treat this as two separate problems

- $x(t) = 3 \cos(4t) + 5 \sin(4t)$
- $x(t) = 6 \cos(10t)$

The total input is the sum of the two $x(t)$'s. The total output is the sum of the two $y(t)$'s

$$y(t) = -2.120 \cos(4t) + 0.040 \sin(4t) \\ -0.404 \cos(10t) + 0.202 \sin(10t)$$

Solving Transfer Functions with Step Inputs

There are several ways to do this. My preference is to use a table

Common LaPlace Transforms		
Name	Time: $y(t)$	LaPlace: $Y(s)$
delta (impulse)	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
	$a \cdot e^{-bt}u(t)$	$\frac{a}{s+b}$
	$2a \cdot e^{-bt} \cos(ct - \theta)u(t)$	$\left(\frac{a \angle \theta}{s+b+jc}\right) + \left(\frac{a \angle -\theta}{s+b-jc}\right)$

Example: Find the impulse response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)X$$

$$Y = \left(\frac{5}{s+3}\right)(1)$$

From the above table:

$$y(t) = 5e^{-3t}u(t)$$

Example: Find the step response of

$$G(s) = \left(\frac{5}{s+3}\right)$$

Solution: Translating:

$$Y = \left(\frac{5}{s+3}\right)\left(\frac{1}{s}\right) = \left(\frac{5}{s(s+3)}\right)$$

Here we have a problem: the above table doesn't include this type of function. Using partial fractions, however, you can turn it into something that is in the table

$$\left(\frac{5}{s(s+3)}\right) = \left(\frac{A}{s}\right) + \left(\frac{B}{s+3}\right)$$

Near $s = 0$, both sides go to infinity. The 'B' term is insignificant near $s=0$, so ignore it. Then

$$\left(\frac{5}{s(s+3)}\right)_{s \rightarrow 0} = \left(\frac{A}{s}\right)_{s \rightarrow 0}$$

Clearing out the 's' terms

$$\left(\frac{5}{s+3}\right)_{s \rightarrow 0} = A$$

$$A = 5/3$$

When s approaches -3 , both sides again go to infinity. The 'A' term is finite and insignificant in this case, meaning

$$\left(\frac{5}{s(s+3)}\right)_{s \rightarrow -3} = \left(\frac{B}{s+3}\right)_{s \rightarrow -3}$$

Cancelling the $(s+3)$ terms

$$\left(\frac{5}{s}\right)_{s \rightarrow -3} = B$$

$$B = -5/3$$

So,

$$Y = \left(\frac{5}{s(s+3)}\right) = \left(\frac{5/3}{s}\right) - \left(\frac{5/3}{s+3}\right)$$

Using the above table for each term:

$$y(t) = \left(\frac{5}{3} - \frac{5}{3}e^{-3t}\right)u(t)$$

Solving with Repeated Roots:

Option #1: Use partial fractions and an expanded table.

Exampe: Find the inverse LaPlace tranform for

$$Y(s) = \left(\frac{4}{(s+2)^2}\right)\left(\frac{1}{s}\right)$$

This expands as

$$Y(s) = \left(\frac{A}{s}\right) + \left(\frac{B}{(s+2)^2}\right) + \left(\frac{C}{s+2}\right)$$

Using the cover-up method

$$A = \left(\left(\frac{4}{(s+2)^2}\right)\left(\frac{1}{s}\right)\right)_{s=0} = 1$$

$$B = \left(\left(\frac{4}{s}\right)\left(\frac{1}{s}\right)\right)_{s=-2} = -2$$

$$C = \left(\frac{d}{ds}\left(\frac{4}{s}\right)\right)_{s=-2} = \left(\frac{-4}{s^2}\right)_{s=-2} = -1$$

giving

$$Y(s) = \left(\frac{1}{s}\right) + \left(\frac{-2}{(s+2)^2}\right) + \left(\frac{-1}{s+2}\right)$$

Now use a table to find the inverse transform of each term

$$y(t) = (1 - 2t e^{-2t} - e^{-2t})u(t)$$

Option #2: Change the problem so that the roots are no longer repeated. From a practical standpoint, there is almost no difference between

$$Y(s) = \left(\frac{4}{(s+2)(s+2)}\right)\left(\frac{1}{s}\right)$$

and

$$Y(s) = \left(\frac{4}{(s+2.01)(s+1.99)}\right)\left(\frac{1}{s}\right)$$

In the latter case, the roots are no longer repeated.

Solving with Complex Roots:

If you don't mind complex numbers, complex roots are no different than real roots: you just wind up with complex numbers in the partial fraction expansion.

The relevant entry in the LaPlace Transform table is

$$\left(\frac{a\angle\theta}{s+b+jc}\right) + \left(\frac{a\angle-\theta}{s+b-jc}\right) \Rightarrow 2a \cdot e^{-bt} \cos(ct - \theta)u(t)$$

Example: Find the $y(t)$ given that

$$Y(s) = G \cdot U = \left(\frac{15}{s^2+2s+10}\right) \cdot \left(\frac{1}{s}\right)$$

Solution: Factoring $Y(s)$

$$Y(s) = \left(\frac{15}{(s)(s+1+j3)(s+1-j3)}\right)$$

Using partial fraction expansion:

$$Y(s) = \left(\frac{1.5}{s}\right) + \left(\frac{0.7906\angle-161.56^0}{s+1+j3}\right) + \left(\frac{0.7906\angle161.56^0}{s+1-j3}\right)$$

$$y(t) = 1.5 + 1.5812 \cdot e^{-t} \cdot \cos(3t + 161.56^0) \quad \text{for } t > 0$$

Properties of LaPlace Transforms

Properties of LaPlace Transforms

Let's start with some properties of LaPlace transforms:

Linearity: $af(t) + bg(t) \Leftrightarrow aF(s) + bG(s)$

Convolution: $f(t) * g(t) \Leftrightarrow F(s) \cdot G(s)$

Differentiation: $\frac{dy}{dt} \Leftrightarrow sY - y(0)$

$$\frac{d^2y}{dt^2} \Leftrightarrow s^2Y - sy(0) - \frac{dy(0)}{dt}$$

Integration: $\int_0^t x(\tau)d\tau = \frac{1}{s}X(s)$

Delay $x(t - T) \Leftrightarrow e^{-sT}X(s)$

Proofs:

Linearity:

$$\begin{aligned} L(af(t) + bg(t)) &= \int_{-\infty}^{\infty} (af(t) + bg(t)) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} (af(t)) \cdot e^{-st} \cdot dt + \int_{-\infty}^{\infty} (bg(t)) \cdot e^{-st} \cdot dt \\ &= a \int_{-\infty}^{\infty} f(t) \cdot e^{-st} \cdot dt + b \int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \\ &= aF(s) + bG(s) \end{aligned}$$

Convolution:

$$\begin{aligned} f(t) * g(t) &= \int_{-\infty}^{\infty} f(t - \tau) \cdot g(\tau) \cdot d\tau \\ L(f(t) * g(t)) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t - \tau) \cdot g(\tau) \cdot d\tau \right) \cdot e^{-st} \cdot dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t - \tau) \cdot g(\tau) \cdot e^{-st} \cdot dt \right) \cdot d\tau \end{aligned}$$

$$= \left(\int_{-\infty}^{\infty} f(t - \tau) \cdot e^{-st} \cdot dt \right) \cdot \left(\int_{-\infty}^{\infty} g(t) \cdot e^{-st} \cdot dt \right)$$

$$= F(s) \cdot G(s)$$

Differentiation:

$$L\left(\frac{dx}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Assume causal (zero for $t < 0$)

$$L\left(\frac{dx}{dt}\right) = \int_0^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\int a' \cdot b \cdot dt = ab - \int a \cdot b' \cdot dt$$

Let

$$a' = \frac{dx}{dt}$$

$$a = x$$

$$b = e^{-st}$$

then

$$L\left(\frac{dx}{dt}\right) = \int_0^{\infty} \left(\frac{dx(t)}{dt}\right) \cdot e^{-st} \cdot dt$$

$$= (x \cdot e^{-st})_0^{\infty} - \int_{-\infty}^{\infty} -s \cdot x(t) \cdot e^{-st} \cdot dt$$

$$= -x(0) + s \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= sX - x(0)$$

Integration:

$$L\left(\int_0^t x(\tau) \cdot d\tau\right) = \int_{-\infty}^{\infty} \left(\int_0^t x(\tau) \cdot d\tau\right) \cdot e^{-st} \cdot dt$$

Integrate by parts.

$$\int a \cdot b' \cdot dt = ab - \int a' \cdot b \cdot dt$$

Let

$$a = \int_0^t x(\tau) \cdot d\tau$$

$$b' = e^{-st}$$

then

$$a' = x$$

$$b = \frac{-1}{s} e^{-st}$$

$$\begin{aligned} L\left(\int_0^t x(\tau) \cdot d\tau\right) &= \int_{-\infty}^{\infty} \left(\int_0^t x(\tau) \cdot d\tau\right) \cdot e^{-st} \cdot dt \\ &= \left(\int_0^t x(\tau) \cdot d\tau \cdot \frac{-1}{s} e^{-st}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x \cdot \frac{-1}{s} e^{-st} \cdot dt \end{aligned}$$

Assuming the function vanishes at infinity

$$\begin{aligned} &= \frac{1}{s} \int_{-\infty}^{\infty} x \cdot dt \\ &= \left(\frac{1}{s}\right) X(s) \end{aligned}$$

Time Delay

$$L(x(t-T)) = \int_{-\infty}^{\infty} x(t-T) \cdot e^{-st} \cdot dt$$

Do a change of variable

$$t - T = \tau$$

$$\begin{aligned} L(x(t-T)) &= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s(\tau+T)} \cdot d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot e^{-sT} \cdot d\tau \\ &= e^{-sT} \cdot \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot d\tau \\ &= e^{-sT} \cdot X(s) \end{aligned}$$