Uniform Distribution

Definitions:

- Uniform Distribution: The probability of each valid outcome is the same.
- Geometric Distribution: The number of Bernoulli trials until you get a success
- Pascal Distribution: The number of Bernoulli trials until you get r successes

distribution	description	pdf	mgf	mean	variance
Uniform range = (a,b)	toss an n-sided die	$\left(\frac{1}{1+b-a}\right)\sum_{k=a}^{b}\delta(k-m)$	$\left(\frac{1}{n}\right)\left(\frac{1+z+z^2+...+z^{n-1}}{z^{n-1}}\right)$	$\left(\frac{a+b}{2}\right)$	$\left(\frac{(b-a+1)^2-1}{12}\right)$
Geometric	Bernoulli until 1st success	$(1-p)^{k-1}p$	$\left(\frac{(1-p)z}{z-p}\right)$	$\left(\frac{1}{p}\right)$	$\left(\frac{1-p}{p^2}\right)$
Pascal	Bernoulli until rth success	$\binom{k+r-1}{k}(1-p)^r p^k$	$\left(\frac{(1-p)z}{z-p}\right)^r$	$\left(\frac{r}{p}\right)$	$\left(\frac{r(1-p)}{p^2}\right)$

source: Wikipedia

Uniform Distribution:

A uniform distribution can be thought of as an extension of a Bernoulli trial where

- There are n possible outcomes (rather than just two), and
- All outcomes have the same probability.

In general the pdf for a uniform distribution over the range of (a, b) (where n = 1+b-a = the number of possible outcomes) is

$$X(m) = \left(\frac{1}{n}\right)\sum_{k=a}^{b}\delta(k-m)$$

The z-transform for a uniform distribution over the range of [a, b] is

$$X(z) = \left(\frac{1}{6}\right)\left(\frac{1+z+z^2+z^3+z^4+...+z^{b-a}}{z^b}\right)$$

The mean is the average:

$$\bar{x} = \left(\frac{a+b}{n}\right)$$

and the variance is approximately

$$s^2 \approx \left(\frac{(b-a)^2}{12}\right)$$

There are many examples of uniform distributions, such as

- Drawing a card from a deck of cards (each has a probability of 1/52)
- A number coming up in Roulette (1 in 31 in Vegas, 1 in 32 in Atlantic City)
- A number coming up in the lottery (1 in 78,960,960)
- Being selected for jury duty (1 in 15,000. Ten people are selected from a county population of 150,000)

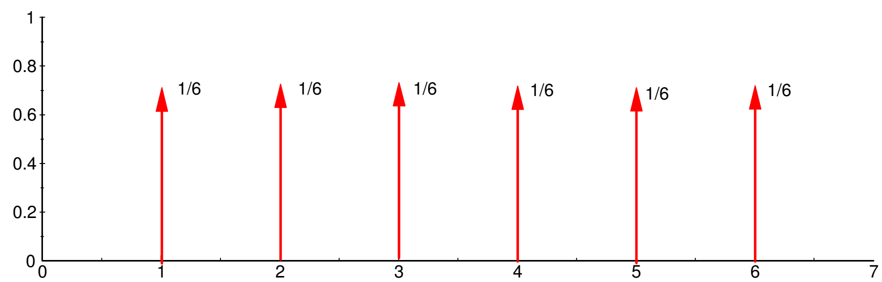
There are some betting schemes which take advantage of processes which are supposed to be uniform but are not. For example, about 15 years ago, someone watched which numbers came up in Roulette in Vegas and found that some numbers were more common that others. He/she (I forget which) won money with this scheme. Now, the roulette wheels are mixed every night (under tight security) and you are not allowed to watch and take notes.

6-Sided Die

For example, a fair six sided die would have six possible outcomes, each with probability of 1/6

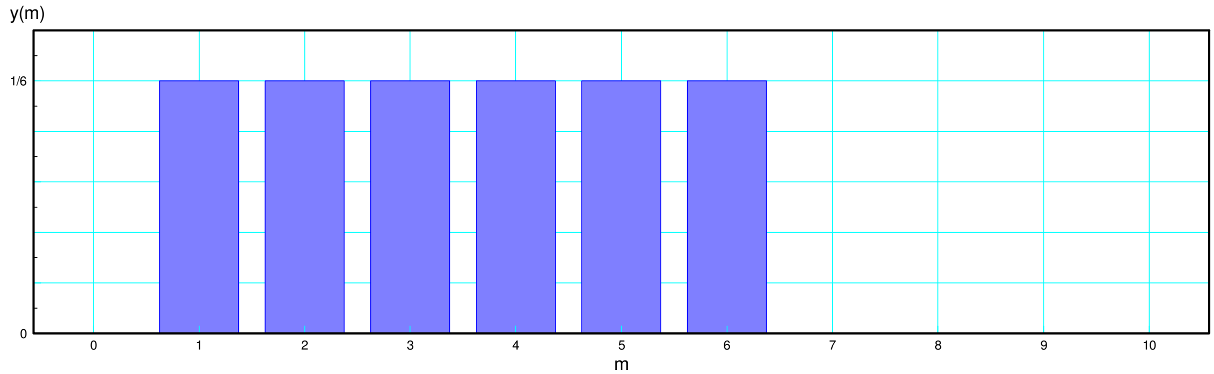
Die Roll: m	0	1	2	3	4	5	6	7+
Probability: X(m)	0	1/6	1/6	1/6	1/6	1/6	1/6	0

The pdf for this would be a delta function at each integer value:



pdf for a fair 6-sided die

Often times, this is represented using a bar graph with the understanding that the pdf is only non-zero at the integer values



The mean of a fair 6-sided die (d6) is

$$\bar{x} = \left(\frac{1+2+3+4+5+6}{6}\right) = 3.5$$

The variance is

$$s^2 = \left(\frac{(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)}{6}\right)$$

$$s^2 = 2.91667$$

Two 6-sided die (2d6)

The sum of two 6-sided die is the convolution of the two pdf's. With a fair 6-sided die (scaled by 6x), the pdf is:

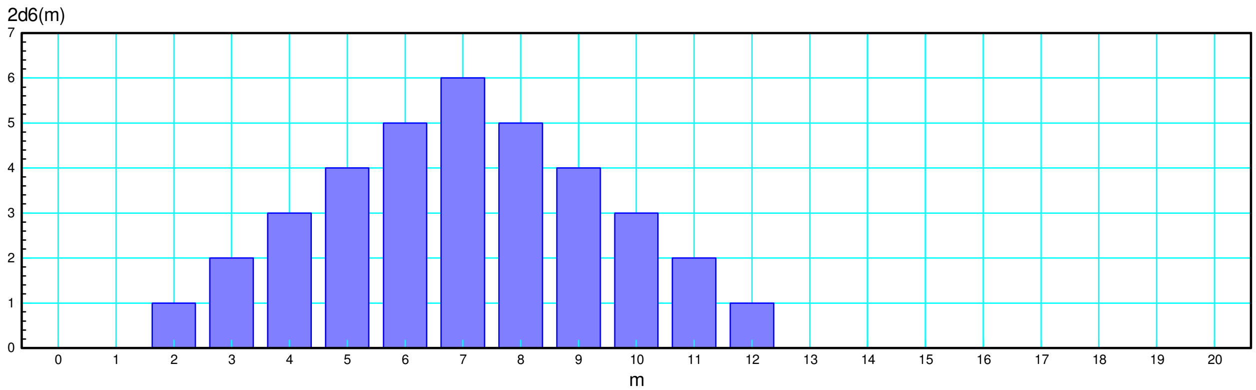
```
d6 = [0,1,1,1,1,1,1]

d6 =      0      1      1      1      1      1      1
```

Two six-sided dice will be (scaled by $6^2 = 36$)

```
d6x2 = conv(d6,d6)

d6x2 =    0    0    1    2    3    4    5    6    5    4    3    2    1
```



pdf of rolling 2d6 (scaled by 36)

The mean is

$$\bar{x} = 2 \cdot 3.5 = 7$$

The variance is

$$s^2 = 2 \cdot 2.9167 = 5.8333$$

You can check this in Matlab by rolling two dice 100,000 times. Start with rolling two dice:

```
for i=1:10
   DICE = ceil(6*rand(1,2));
   disp([DICE, sum(DICE)])
end
```

A	B	A+B
5	1	6
2	2	4
5	1	6
4	6	10
5	2	7
1	5	6
4	4	8
4	3	7
1	3	4

Now place the sum into an array and take the mean and standard deviation

```
X = zeros(1e5, 1);

for i=1:1e5
   DICE = ceil(6*rand(1,2));
   X(i) = sum(DICE);
end

x = mean(X)
s2 = std(X)^2
```

```
x =     7.0053       vs. 7.0000

s2 =    5.8375       vs. 5.8333
```

Problem: Determine the probability of rolling 10 or higher with 2d6.

Solution: The number of combinations that result in rolls of 10 or higher are

- 10: frequency = 3
- 11: frequency = 2
- 12: frequency = 1

so the probability is

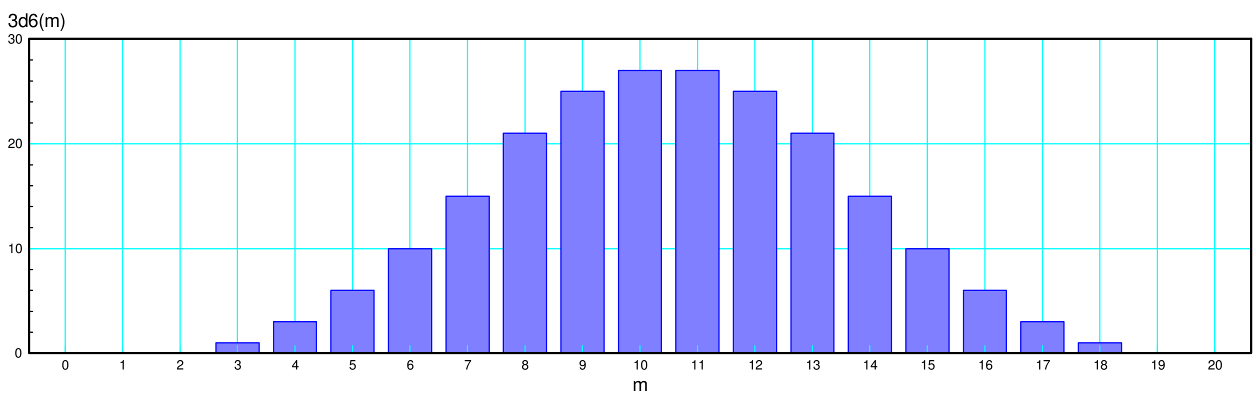
$$p(m \geq 10) = \left(\frac{3+2+1}{36}\right) = \frac{6}{6}$$

Three six-sided dice (3d6)

Scaling the pdf of a 6-sided die by 6 results in the pdf scaled by 6^3 (216)

```
d6x3 = conv(d6x2, d6)

m    0    1    2    3    4    5    6    7    8    9   10   11   12   13   14   15   16   17   18
     0    0    0    1    3    6   10   15   21   25   27   27   25   21   15   10    6    3    1
```



pdf of rolling d6 (scaled by 216)

Note that this is quickly converging to a bell-shaped curve (a normal distribution - i.e. the Central Limit Theorem again).

The probability of rolling 16 or higher is thus

$$p(m \geq 16) = \left(\frac{6+3+1}{216}\right) = \left(\frac{10}{216}\right)$$

The mean is

$$\bar{x} = 3 \cdot 3.5 = 10.5$$

The variance is

$$s^2 = 3 \cdot 2.9167 = 8.75$$

You can check this in Matlab by rolling three dice 100,000 times. Start with rolling three dice:

```
for i=1:10
   DICE = ceil(6*rand(1,3));
   disp([DICE, sum(DICE)])
end
```

A	B	C	A+B+C
4	4	2	10
4	6	2	12
1	3	3	7
6	3	5	14
2	5	3	10
4	5	3	12
5	4	5	14
5	2	3	10
2	3	5	10
3	5	5	13

The mean and variance are found from

```
X = zeros(1e5, 1);

for i=1:1e5
   DICE = ceil(6*rand(1,3));
   X(i) = sum(DICE);
end

x = mean(X)
s2 = std(X)^2
```

resulting in

```
x =   10.5015           vs. 10.50


s2 =    8.6896          vs. 8.75
```

As the sample size goes to infinity, these converge to the theoretical mean and variance.