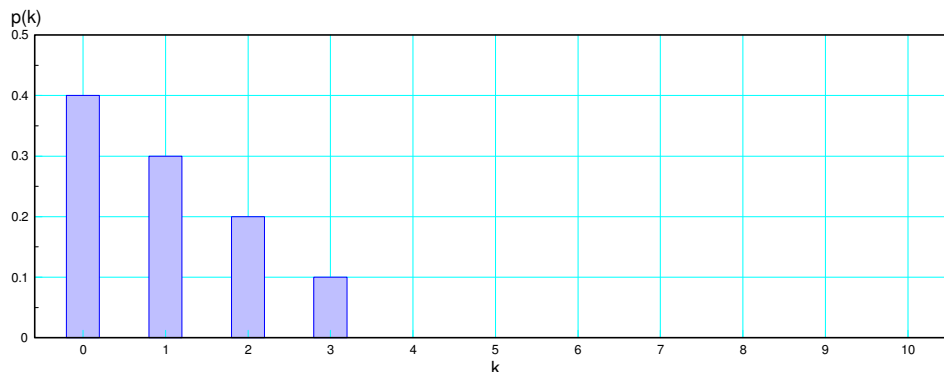


## z-Transforms



Discrete pdf with a z-transform of  $\psi(z) = 0.4 + 0.3\left(\frac{1}{z}\right) + 0.2\left(\frac{1}{z^2}\right) + 0.1\left(\frac{1}{z^3}\right)$

### z-Transform:

z-transforms are similar to Laplace transforms - only they apply to discrete-time systems (or discrete probability density functions). The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Hence, the above figure which has the pdf of

$$pdf = 0.4 \delta(k) + 0.3 \delta(k-1) + 0.2 \delta(k-2) + 0.1 \delta(k-3)$$

has a z-transform (also known as the moment generating function) of

$$\psi(z) = 0.4 + 0.3\left(\frac{1}{z}\right) + 0.2\left(\frac{1}{z^2}\right) + 0.1\left(\frac{1}{z^3}\right)$$

The assumption behind the z-transform is that all functions are in the form of

$$y(k) = z^k$$

When you move forward in time by one sample, you multiply by 'z'.

$$y(k+1) = z^{k+1} = z \cdot z^k = z \cdot y(k)$$

'zy' is read as 'the next value of y'. Likewise, the z-transform converts difference equations into algebraic equations in 'z'.

## z-Transform Properties: ([www.wikipedia.com](http://www.wikipedia.com))

The z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}$$

Properties:

### Linearity:

$$Z(ax_n + by_n) = aX(z) + bY(z)$$

Proof: The z-transform is

$$\begin{aligned} Z(ax_n + by_n) &= \sum_{n=-\infty}^{\infty} (ax_n + by_n) \cdot z^{-n} \\ Z(ax_n + by_n) &= \left( a \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} \right) + \left( b \sum_{n=-\infty}^{\infty} y_n \cdot z^{-n} \right) \\ Z(ax_n + by_n) &= aX(z) + bY(z) \end{aligned}$$

### Time Shifting:

$$Z(x_{n-k}) = z^{-k} \cdot X(z)$$

Proof:

$$Z(x_{n-k}) = \sum_{n=-\infty}^{\infty} x_{n-k} \cdot z^{-n}$$

Let  $m = n-k$

$$\begin{aligned} Z(x_{n-k}) &= \sum_{m=-\infty}^{\infty} x_m \cdot z^{-(m+k)} \\ Z(x_{n-k}) &= \sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \cdot z^{-k} \\ Z(x_{n-k}) &= z^{-k} \cdot \left( \sum_{m=-\infty}^{\infty} x_m \cdot z^{-m} \right) \\ Z(x_{n-k}) &= z^{-k} \cdot X(z) \end{aligned}$$

Multiplying by  $1/z$  means delay the signal by one.

### Convolution:

$$Z(x_n * y_n) = X(z) \cdot Y(z)$$

Proof:

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$$Z\left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n}$$

Change the order of summation:

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_k \cdot y_{n-k}\right) \cdot z^{-n} \\ &= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_{n-k}\right)\right) \cdot z^{-n} \end{aligned}$$

Let  $m = n-k$

$$\begin{aligned} &= \left(\sum_{k=-\infty}^{\infty} x_k \left(\sum_{n=-\infty}^{\infty} y_m\right)\right) \cdot z^{-(m+k)} \\ &= \left(\left(\sum_{k=-\infty}^{\infty} x_k \cdot z^{-k}\right) \left(\sum_{n=-\infty}^{\infty} y_m \cdot z^{-m}\right)\right) \\ &= X(z) \cdot Y(z) \end{aligned}$$

Convolution is the biggie - it turns convolution into multiplication.

**Table of z-Transforms:**

function	y(k) (k > 0)	Y(z)
delta	$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & \textit{otherwise} \end{cases}$	1
unit step	$u(k) = 1$	$\left(\frac{z}{z-1}\right)$
ramp	$k$	$\left(\frac{z}{(z-1)^2}\right)$
parabola	$k^2$	$\left(\frac{z(z+1)}{(z-1)^3}\right)$
cubic	$k^3$	$\left(\frac{z(z^2+4z+1)}{(z-1)^4}\right)$
decaying exponential	$a^k$	$\left(\frac{z}{z-a}\right)$
	$\left(\frac{k}{1!}\right) a^{k-1}$	$\left(\frac{z}{(z-a)^2}\right)$
	$\left(\frac{k(k-1)}{2!}\right) a^{k-2}$	$\left(\frac{z}{(z-a)^3}\right)$
	$\left(\frac{k(k-1)(k-2)}{3!}\right) a^{k-3}$	$\left(\frac{z}{(z-a)^4}\right)$
damped sinewave	$2b \cdot a^k \cdot \cos(k\theta + \phi) \cdot u(k)$	$\left(\frac{(b\angle\phi)z}{z-(a\angle\theta)}\right) + \left(\frac{(b\angle-\phi)z}{z-(a\angle-\theta)}\right)$

Proof: Delta Function. This is sort-of the definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} = \dots + x_0 \cdot z^0 + x_1 \cdot z^1 + x_2 \cdot z^2 + \dots$$

If x(k) is a delta function:

$$X(z) = 1$$

Proof: Unit Step. Using a table:

	$z^2$	$z^1$	$z^0$	$z^{-1}$	$z^{-2}$	$z^{-3}$	$z^{-4}$
X(z)	0	0	1	1	1	1	1
$z^{-1} X(z)$	0	0	0	1	1	1	1
subtract							
$\left(1 - \frac{1}{z}\right) X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \frac{1}{\left(1 - \frac{1}{z}\right)} = \left(\frac{z}{z-1}\right)$$

Proof: Decaying Exponential. Using a table:

	$z^2$	$z^1$	$z^0$	$z^{-1}$	$z^{-2}$	$z^{-3}$	$z^{-4}$
$X(z)$	0	0	1	a	$a^2$	$a^3$	$a^4$
$a * z^{-1} X(z)$	0	0	0	a	$a^2$	$a^3$	$a^4$
subtract							
$\left(1 - \frac{a}{z}\right)X(z)$	0	0	1	0	0	0	0

so

$$X(z) = \left(\frac{1}{1 - \frac{a}{z}}\right) = \left(\frac{z}{z-a}\right)$$

### Solving Functions in the z-Domain

Problem 1: Find the step response of

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)X$$

i) Replace  $X(z)$  with the z-transform of a step

$$Y = \left(\frac{0.2z}{(z-0.9)(z-0.5)}\right)\left(\frac{z}{z-1}\right)$$

ii) Use partial fractions to expand this. Note, however, that the table entries have a 'z' in the numerator. So, factor this out first then take the partial fraction expansion

$$Y = \left(\frac{0.2z}{(z-1)(z-0.9)(z-0.5)}\right)z$$

$$Y = \left(\left(\frac{4}{z-1}\right) + \left(\frac{-4.5}{z-0.9}\right) + \left(\frac{0.5}{z-0.5}\right)\right)z$$

Multiply through by z

$$Y = \left(\left(\frac{4z}{z-1}\right) + \left(\frac{-4.5z}{z-0.9}\right) + \left(\frac{0.5z}{z-0.5}\right)\right)$$

iii) Now apply the table entries

$$y(k) = 4 - 4.5 \cdot (0.9)^k + 0.5 \cdot (0.5)^k \quad k \geq 0$$

Problem 2: Find the step response of a system with complex poles:

$$Y = \left( \frac{0.2z}{(z-0.9\angle 10^0)(z-0.9\angle -10^0)} \right) X$$

i) Replace X with its z-transform (a unit step)

$$Y = \left( \frac{0.2z}{(z-0.9\angle 10^0)(z-0.9\angle -10^0)} \right) \left( \frac{z}{z-1} \right)$$

ii) Factor our a z and use partial fractions

$$Y = \left( \left( \frac{5.355}{z-1} \right) + \left( \frac{2.98\angle 153.97^0}{z-0.9\angle 10^0} \right) + \left( \frac{2.98\angle -153.97^0}{z-0.9\angle -10^0} \right) \right) z$$

iii) Convert back to time using the table of z-transforms

$$(k) = 5.355 + 4.859 \cdot (0.9)^k \cdot \cos(10^0 \cdot k - 153.97^0) \quad k \geq 0$$

Note that in the s-domain, rectangular coordinates are more convenient

- The real part of s tells you the rate at which the exponential decays
- The complex part of s tells you the frequency of oscillations.

In the z-domain, polar coordinates are more convenient

- The amplitude of z tells you the rate at which the signal decays
- The angle of z tells you the frequency of oscillation.

In this case,

- The signal decays by 10% each sample  $(0.9)^k$
- The phase changes by 10 degrees each sample, or requires 36 samples per cycle

## Time Value of Money

As a sidelight, you can also solve time-value of money problems using z-transforms.

Assume you borrow \$100,000 for a house. How much do you have to pay each month to pay off the loan in 10 years? Assume 6% interest per year (0.5% per month).

Solution: Let  $x(k)$  be how much money you owe today. The amount you owe next month,  $x(k+1)$ , is

$$x(k+1) = 1.005x(k) - p + X(0) \cdot \delta(k)$$

where 'p' is your monthly payment. Converting to the z-domain

$$zX - X(0) = 1.005X - p \left( \frac{z}{z-1} \right)$$

$$(z - 1.005)X = X(0) - p\left(\frac{z}{z-1}\right)$$

$$X = \left(\frac{X(0)}{z-1.005}\right) - p\left(\frac{z}{(z-1)(z-1.005)}\right)$$

Using partial fractions

$$X = \left(\frac{X(0)}{z-1.005}\right) + pz\left(\left(\frac{200}{z-1}\right) - \left(\frac{200}{z-1.005}\right)\right)$$

Converting back to the time domain

$$x(k) = 1.005^k X(0) - 200p(1.005^k - 1)u(k)$$

After 10 years (k=120 payments), x(k) should be zero

$$x(120) = 0 = \$181,939 - 200p(0.8194)$$

$$p = \$1110.20$$

Your monthly payments are \$1,110.20.

If you stretch this out to 30 years (k = 360 payments), the monthly payment becomes

$$x(360) = 0 = \$602,257 - 200p(5.0226)$$

$$p = \$599.55$$

Paying off the loan over a time span 3 times longer doesn't reduce the payments by 3 times. It's actually only 46% less. The total amount you'll pay on the loan, however, increases from \$133,224 to \$215,838.

note: That's pretty much all a business calculator is: a calculator which does z-transforms where the keys are renamed "interest rate", "initial loan value" and "number of payments."

