# 3. Conditional Probability: 5-Card Draw

## Poker: 5-Card Draw

With 5-card draw

- You draw 5 cards to start
- Betting then commences.
- You then discard N cards and draw N new cards.
- Betting then continues
- Cards are revealed and the highest hand wins.

With 5-card draw, the odds of getting different hands is harder to compute since the number of cards you are dealt depends upon the number of cards you draw. The number of cards you draw depends upon the hand you have.

This is where conditional probabilities come into play

$$p(A) = p(A|B)p(B)$$

for all possibilities B.

As a reminder, with the first 5 cards you draw, the odds of a given hand are:

| POKER HAND | | COUNT | Odds Against |
|---|---|---|---|
| 2 | Straight Flush | 40 | 64,974 |
| 3 | Four of a Kind | 624 | 4,165 |
| 4 | Full House | 3,744 | 694.17 |
| 5 | Flush | 5,108 | 508.8 |
| 6 | Straight | 10,200 | 254.8 |
| 7 | Three of a Kind | 54,912 | 47.33 |
| 8 | Two Pair | 123,552 | 21.04 |
| 9 | One Pair | 1,098,240 | 2.37 |
| 10 | High Card | 1,302,540 | 2 |
| Total | | 2,598,960 | |

## Four-of-a-Kind:

Assume there are four ways you can end up with a 4-of-a-kind:

Start with

- A: 4-of-a-kind (and do nothing)
- B: 3-of-a-kind (draw 2 new cards)
- C: Pair (draw 3 new cards)
- D: High-Card hand (draw 5 new cards)

The probability of getting a 4-of-a-kind is

$$p(x) = P(x|A)p(A) + p(x|B)p(B) + p(x|C)p(C) + p(x|D)p(D)$$

A: Starting with 4-of-a-kind

The probability of ending up with a 4-of-a-kind is 1.000

$$p(x|A) = 1.000$$

B: Starting with 3-of-a-kind (and draw 2 cards)

The number of ways you can draw 2 cards is

$$M = \binom{47}{2} = 1,081$$

The number of ways you can end up with a 4-of-a-kind is

- Of the one remaining card that matches the cards you keep, choose 1 (1 choose 1)
- Of the 46 remaining cards in the deck, choose 1 (46 choose 1)

$$N = \binom{1}{1}\binom{46}{1} = 46$$

The odds are then

$$p(x|B) = \left(\frac{46}{1081}\right) = \frac{1}{23.5}$$

Another way to compute this is the probability of getting a 4-of-a-kind is 1 minus the probability of not getting 4 of a kind. To fail to get a 4-of-a-kind

- The first card has to be a different value (46/47 chance)
- The second card also has to be different (45/46 chance)

$$p(x|B) = 1 - \left(\frac{46}{47}\right)\left(\frac{45}{46}\right) = \left(\frac{2}{47}\right) = \left(\frac{1}{23.5}\right)$$

C: Starting with a pair, you draw 3 new cards

The number of ways you can draw 3 new cards is

$$M = \binom{47}{3} = 16,215$$

To end up with a 4-of-a-kind

- Of the 2 matching cards in the deck, pick two
- Of the remaining 45 cards, pick one

$$N = \binom{2}{2}\binom{45}{1} = 45$$

The odds of getting a four-of-a-kind is then

$$p(x|C) = \left(\frac{45}{16,215}\right) = \left(\frac{1}{360.33}\right)$$

D: High-Card Hand. Draw 5 new cards. The odds are the same as any new hand

$$p(x|D) = \left(\frac{1}{4,165}\right)$$

Result: The probability of getting 4-of-a-kind with 5-card draw is thus

p(x) = P(x|A)p(A) + p(x|B)p(B) + p(x|C)p(C) + p(x|D)p(D)

$$p(x) = (1.000)\left(\frac{1}{4165}\right) \qquad \textit{dealt 4-of-a-kind}$$

$$+\left(\frac{1}{23.5}\right)\left(\frac{1}{47.33}\right) \qquad \textit{dealt 3-of-a-kind}$$

$$+\left(\frac{1}{360.33}\right)\left(\frac{1}{2.37}\right) \qquad \textit{pair}$$

$$+\left(\frac{1}{4165}\right)\left(\frac{1}{2}\right) \qquad \textit{high-card hand}$$

$$p(x) = 0.002430 = \frac{1}{411.5}$$

With 5-card draw, the odds of getting a 4-of-a-kind increase from 4,165 : 1 against to 411.5 : 1 against

## Monte-Carlo Simulation

Repeat the previous program for 5-card draw. Add a block of code which

- Sorts the cards based upon their frequency (plus 0.01 * card value so the sort gives unique results)
- Keeps all cards that pair up (frequency > 1)
- Replaces all cards which don't have a pair with the next cards in the deck

Once you draw cards, then recompute how many cards match up (what kind of hand it is)

To check the program (next page), the frequency of each hand

- With 0 draws (previous lecture), and
- One draw

is compared to our calculations. These are not exactly the same, but within the right ballpark. It looks like the simulation is correct.

| | 0 Draws 5-Card Stud | | 1-Draw 5-Card Draw | |
|---|---|---|---|---|
| | Simulation | Calculation | Simulation | Calculation |
| 4 of a kind | 17 | 24 | 245 | 243 |
| full house | 152 | 144 | 1,190 | ? |
| 3 of a kind | 2,148 | 2,112 | 7,824 | ? |
| 2-pair | 4,836 | 4,753 | 13,162 | ? |
| 1-pair | 42,102 | 42,194 | 51,682 | ? |
| High-Card | 50,745 | 50,773 | 25,897 | ? |

Number of each type of hand in 100,000 games

With a Monte-Carlo simulation, you can change the game and allow multiple draw rounds. These would be very tedious with conditional probabilities. With a simulation, it's just increase the count size in a for-loop

| | 0 Draws 5-card stud | 1 Draw 5-Card Draw | 2 Draws | 3 Draws |
|---|---|---|---|---|
| 4 of a kind | 17 | 245 | 1,063 | 2,597 |
| full house | 152 | 1,190 | 3,329 | 6,667 |
| 3 of a kind | 2,148 | 7,824 | 13,484 | 18,043 |
| 2-pair | 4,836 | 13,162 | 21,727 | 27,687 |
| 1-pair | 42,102 | 51,682 | 47,619 | 39,211 |
| High-Card | 50,745 | 25,897 | 12,778 | 5,795 |

Number of each type of hand in 100,000 games

Matlab Code

note: This code ignores straights and flushes

```
% 5-Card Draw
% Conditional Probability

N_Draw = 0;

Pair4 = 0;
FullHouse = 0;
Pair3 = 0;
Pair22 = 0;
Pair2 = 0;

TopCard = 1;

for games = 1:1e5

   X = rand(1,52);
   [a,Deck] = sort(X);
   TopCard = 1;
   for i=1:5
       Hand(i) = Deck(TopCard);
       TopCard = TopCard + 1;
      end

for Draw = 0:N_Draw

   Value = mod(Hand,13) + 1;
   Suit = floor(Hand/13) + 1;

   Freq = [1:5] * 0.0001 + Value * 0.01;  % not zero so sort gives unique
result
   for i=1:5
      for j=1:5
          if(Value(i) == Value(j))
              Freq(i) = Freq(i) + 1;
          end
      end
   end

   [a,b] = sort(Freq);
   Value = Value(b);
   Hand = Hand(b);
   Suit = Suit(b);

   N = zeros(1,13);
   for n=1:13
      N(n) = sum(Value == n);
   end

   [N,a] = sort(N);

   if(Draw < N_Draw)

      if (N(13) == 4) % 4 of a kind
         Hand(1) = Deck(TopCard);      % Draw one card
         TopCard = TopCard + 1;
         Pair4 = Pair4 + 1;
```

```
      end
      if ((N(13) == 3)*(N(12) == 2)) % full house.  Do nothing
      end
      if ((N(13) == 3)*(N(12) < 2))  % three of a kind
         Hand(1) = Deck(TopCard);          % draw two cards
         TopCard = TopCard + 1;
         Hand(2) = Deck(TopCard);
         TopCard = TopCard + 1;
      end
      if ((N(13) == 2)*(N(12) == 2)) % 2-pair
         Hand(1) = Deck(TopCard);          % draw one card
         TopCard = TopCard + 1;
      end
      if ((N(13) == 2)*(N(12) < 2))   % one pair
         Hand(1) = Deck(TopCard);          % draw three cards
         TopCard = TopCard + 1;
         Hand(2) = Deck(TopCard);
         TopCard = TopCard + 1;
         Hand(3) = Deck(TopCard);
         TopCard = TopCard + 1;
      end
      if (N(13) < 2)                  % high card
         for i=1:5
            Hand(i) = Deck(TopCard);
            TopCard = TopCard + 1;
         end
      end
   end

end


% One last sort before determining type of hand

   Value = mod(Hand,13) + 1;
   Suit = floor(Hand/13) + 1;

   Freq = [1:5] * 0.0001 + Value * 0.01;  % not zero so sort gives unique
result
   for i=1:5
      for j=1:5
          if(Value(i) == Value(j))
              Freq(i) = Freq(i) + 1;
          end
      end
   end

   [a,b] = sort(Freq);
   Value = Value(b);
   Hand = Hand(b);
   Suit = Suit(b);

   N = zeros(1,13);
   for n=1:13
      N(n) = sum(Value == n);
   end

   [N,a] = sort(N);
```

```
if (N(13) == 4) Pair4 = Pair4 + 1; end
if ((N(13) == 3)*(N(12) == 2)) FullHouse=FullHouse + 1; end
if ((N(13) == 3)*(N(12) < 2)) Pair3 = Pair3 + 1; end
if ((N(13) == 2)*(N(12) == 2)) Pair22 = Pair22 + 1; end
if ((N(13) == 2)*(N(12) < 2)) Pair2 = Pair2 + 1; end

end

[Pair4, FullHouse,Pair3,Pair22,Pair2]'
```