2. Combinations and Permutations: 5-Card Stud

Combinatorics is the study of determining how many ways an event can happen. Assuming all events are equally likely, it also allows you to determine the probability of a certain outcome.

Definitions:

- n! "n factorial" n x (n-1) x (n-2) x ... x 2 x 1
- 0! = 1 Just define zero factorial to be one.
- p(x) "the probability of outcome x"
- $P_{n,m}$ "Permutations of n events taken m at a time". $P_{n,m} = \frac{n!}{(n-m)!}$
- $C_{n,m}$ "Combinations of n events taken m at a time" $C_{n,m} = \frac{n!}{m!\cdot(n-m)!}$
- $\binom{n}{m}$ "n choose m". Another way of writing $C_{n,m}$
- Sample With Replacement: Each sample is of the same population size
- Sample Without Replacement: Each time you sample, the remaining population becomes one smaller

Poker: 5-Card Stud

To give us something concrete to relate to, consider the poker game of 5-card stud. In this game, cards are dealt one at a time:

- Starting out, a deck of 52 cards is shuffled.
- The first card is played face down so that only the player sees this card.
- The second card is played face up so all players can see it. After two cards are played, bets are made.
- Once betting stops, a third card is played face up and betting starts over again.
- Ditto for the 4th card and 5th card.
- Once betting is finished with 5 cards for each player, the face down card is revealed and the winner is determined.

The winning hands (in order) for poker are

- Royal Flush: 10-J-Q-K-A of the same suit
- Straight-Flush: A run of 5 cards in the same suit
- 4 of a kind: Four of your cards have the same value. Ex: J-J-J-J-x
- Full-House: 3 of a kind and a pair. Ex: J-J-J-Q-Q
- Flush: All cards of the same suit
- Straight: A run of 5 cards
- 3 of a kind: Three of your cards match. Ex: J-J-J-x,y
- 2-Pair: Two pairs of cards. J-J-Q-Q-x
- Pair: Two cards match. J-J-x-y-z
- High-Card: Other. No pairs, no straights, no flush.

Factorials:

How many ways are there to shuffle a deck of 52 cards?

Answer:

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.
- etc.

The total number of ways a deck can be shuffled is

$$N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot ...2 \cdot 1$$

This is 52 factorial, written as

$$N = 52!$$

Most calculators have a factorial key

$$N = 8.0658 \cdot 10^{67}$$

Note that this is too many combinations for even a computer to run through.

Permutations: (Order Matters)

The number of ways a given hand can play out is..

- Select the first card from 52 possibilities
- The second card from 51 possibilities
- The third card from 50 possibilities.
- etc.

giving

$$N = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$$

Another way to write this is

$$N = \left(\frac{52!}{47!}\right)$$

$$N = 311,875,200$$

which is the number of permutations of 5 cards selected from 52

$$P_{n,m} = \left(\frac{n!}{(n-m)!}\right)$$

There are 311,875,20 different ways a given hand can play out.

Combinations (order doesn't matter)

When determining who won the hand, the order of the cards doesn't matter. This means that permutations double counts each hand based upon the number of ways you can arrange five cards. Let M be the number of ways to do this. Given 5 cards

- Select the first card from 5 possibilities
- The second card from 4 possibilities
- The third card from 3 possibilities.
- etc.

or

$$M = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$M = 5! = 120$$

The number of hands in poker is then

$$\#hands = \left(\frac{52!}{(52-5)! \cdot 5!}\right) = 2,598,960$$

This number is important: we'll be using it to compute the probability of getting different hands.

Combinations is written as

$$C_{n,m} = \binom{n}{m} = \left(\frac{n!}{(n-m)! \cdot m!}\right)$$

and expressed as "n choose m."

Probability of getting a Royal Flush

Once you know the total number of poker hands possible, you can compute the probability of any given hand as

$$p(x) = \frac{\text{the total number of hands that are x}}{2,598,960}$$

The easiest is a royal flush: there are only four possibilities (a royal flush in spades, hearts, clubs, and diamonds)

$$p(\text{royal flush}) = \left(\frac{4}{2,598,960}\right) = \frac{1}{649,740}$$

The odds against getting a royal flush are 649,740 : 1

Another way to think of this, if you played billions of poker games, you should on average get a royal flush once every 649,740 hands.

4 of a kind:

There are several ways to compute this. One ways is as follows:

Assume the cards in your hand are x-x-x-x-y

- There are 13 cards in a deck: Ace through King. Pick one (13 choose 1)
- Of the four cards of that value, choose four (4 choose 4)
- Of the 48 remaining cards, pick one (48 choose 1)

$$N = \binom{13}{1}\binom{4}{4}\binom{48}{1} = 624$$

There are 624 different hands that give you a 4-of-a-kind. The odds against getting this is then

$$p(N) = \left(\frac{624}{2{,}598{,}960}\right) = \left(\frac{1}{4165}\right)$$

The odds against getting a 4-of-a-king is 4165 : 1

3-of-a-kind 54,912

- Of the 13 value, pick one (13 choose 1)
- Of the four cards of that type, pick three (4 choose 3)
- Of the 48 remaining cards, pick two (48 choose 2)
- Minus the case where the 2 cards chosen match (full house)

$$N = \binom{13}{1}\binom{4}{3}\left(\binom{48}{2} - 72\right) = 54,912$$

The odds against getting three-of-a-kind are

$$p(N) = \left(\frac{54{,}912}{2{,}598{,}960}\right)$$

p() = 47.33 : 1 odds against

Flush

- Of the four suits, choose one (4 choose 1)
- Of the 13 cards in that suit, choose five (13 choose 5)

$$N = \binom{4}{1}\binom{13}{5} = 5,184$$

This includes straight-flushes (N = 40). Removing these gives 5,144 different flushes

The odds against getting a flush are

$$p(N) = \left(\frac{5{,}144}{2{,}598{,}960}\right)$$

p() = 505.2 : 1

Straight

- A straight can start with an Ace (Ace-5) through a 10 (10-J-Q-K-A). That gives 40 starting cards (10 values in 4 suits)
- Of the four cards of the next value, choose one (4 choose 1)
- Of the four cards of the next value, choose one (4 choose 1)

so

$$N = 40 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10,240 \quad \textit{all straignts}$$

This also counts straight-flushes (pick the first card. Once picked, all other cards have to line up in the same suit)

$$N = 40 \cdot \binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1} = 40 \quad \textit{straight flushes}$$

so the total number of straights is

$$N = 10,200$$

giving the probability of a straight being

$$p = \left(\frac{10,240}{2,598,960}\right) = \left(\frac{1}{254.8}\right)$$

The odds against getting a straight are 254.8:1 against

2-Pair

- Of the 13 values (Ace through King), pick two (13 choose 2)
- For the first value, pick two cards of the four in the deck (4 choose 2)
- For the second value, pick two cards of the four in the deck (4 choose 2)
- For the last card, pick one (44 choose 1)

$$N = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$$

$$N = 123,552$$

The odds of getting 2-pair are

$$p(x) = \left(\frac{123,552}{2,598,960}\right) = \left(\frac{1}{21.035}\right)$$

or 21.03 : 1 odds against getting two-pair.

Probability of all poker hands

https://allmathconsidered.wordpress.com/2017/05/23/the-probabilities-of-poker-hands/

POKER HAND		COUNT	Odds Against
2	Straight Flush	40	64,974
3	Four of a Kind	624	4,165
4	Full House	3,744	694.17
5	Flush	5,108	508.8
6	Straight	10,200	254.8
7	Three of a Kind	54,912	47.33
8	Two Pair	123,552	21.04
9	One Pair	1,098,240	2.37
10	High Card	1,302,540	2
Total		2,598,960	

Monte-Carlo Simulations

Monte-Carlo simulations are another way to determine probabilities. Here,

- A program is written to generate a random hand of poker.
- The type of hand it is then logged
- This experiment is then repeated a lage number of times (1 million plus)

The odds are then approximately the number of times each hand occurred divided by the number of hands dealt.

Shuffle the deck and draw 5 cards:

```
X = rand(1,52);
[a,Deck] = sort(X);
Deck(1:10)

ans =     47     40     39      3     44     16     17     49      2     25
```

The value of the card is the number mod 13, the suit is the value / 4

```
Hand = Deck(1:5)

Hand =     47     40     39      3     44

Value = mod(Hand, 13) + 1

Value =      9      2      1      4      6

Suit = floor(Hand/13) + 1

Suit =      4      4      4      1      4
```

To determine the number of pairs (2-of-a-kind) etc, count how many times the value of a given card is 1 (Ace), 2, 3, 4, 5, etc

```
N = zeros(1,13);
for n=1:13
    N(n) = sum(Value == n);
end
```

Then determine what kind of hand by

- Sorting the number of matching cards
- Checking if the maximum of the matches is 4 (4-of-a-kind), 3 and 2 (full house), etc

```
[N,a] = sort(N);

if (N(13) == 4) Pair4 = Pair4 + 1; end
if ((N(13) == 3)*(N(12) == 2)) FullHouse=FullHouse + 1; end
if ((N(13) == 3)*(N(12) < 2)) Pair3 = Pair3 + 1; end
if ((N(13) == 2)*(N(12) == 2)) Pair22 = Pair22 + 1; end
if ((N(13) == 2)*(N(12) < 2)) Pair2 = Pair2 + 1; end
```

The results for 1 million hands of poker:

```
          4-of-kind  full-hose    3-o-k        2-pair     pair
X =         252        1409       21097        47498      423055
```

The odds are

```
format long
1e6 ./ X

   3968   709.7   47.4   21.05   2.36

>>
```

Hand	Computed Odds	Simulated Odds
4 of a kind	4165 : 1	3968 : 1
Full-House	694 : 1	709.7 : 1
3 of a kind	47.3 : 1	47.4 : 1
2-pair	21.0 : 1	21.0 : 1
pair	2.37 : 1	2.36 :1

Note: This is essentially the definition of probability: If you repeat an event a large number of times, the occurrences should be the probability times the number of events.

Matlab Code:

```
Pair4 = 0;
FullHouse = 0;
Pair3 = 0;
Pair22 = 0;
Pair2 = 0;

for i0 = 1:1e6

X = rand(1,52);
[a,Deck] = sort(X);
Hand = Deck(1:5);
Value = mod(Hand,13) + 1;
Suit = floor(Hand/13) + 1;

N = zeros(1,13);
for n=1:13
    N(n) = sum(Value == n);
end

[N,a] = sort(N);

if (N(13) == 4) Pair4 = Pair4 + 1; end
if ((N(13) == 3)*(N(12) == 2)) FullHouse=FullHouse + 1; end
if ((N(13) == 3)*(N(12) < 2)) Pair3 = Pair3 + 1; end
if ((N(13) == 2)*(N(12) == 2)) Pair22 = Pair22 + 1; end
if ((N(13) == 2)*(N(12) < 2)) Pair2 = Pair2 + 1; end

end

[Pair4, FullHouse,Pair3,Pair22,Pair2]
```