Combinatorics:

Tree Analysis and Enumeration

ECE 341: Random Processes

Lecture #1

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note: All lecture notes, homework sets, and solutions are posted on www.BisonAcademy.com

Combinatorics

Combinatorics is the study of determining how many ways an event can happen. Assuming all events are equally likely, it also allows you to determine the probability of a certain outcome.

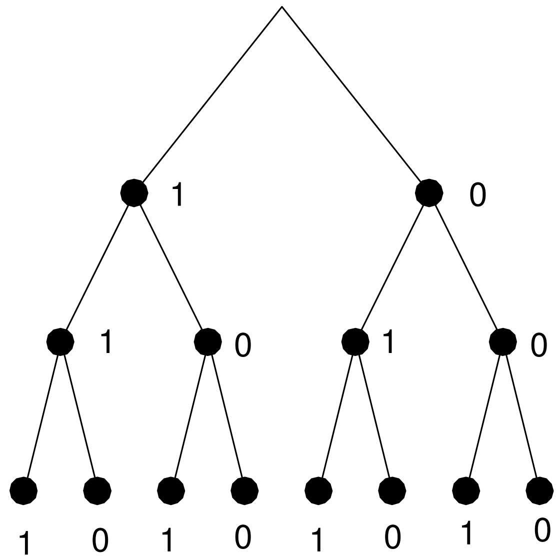
Definitions:

- n! "n factorial" n x (n-1) x (n-2) x ... x 2 x 1
- 0! = 1 Just define zero factorial to be one.
- p(x) "the probability of outcome x"
- ${}_nP_m$ "Permutations of n events taken m at a time". ${}_nP_m = \frac{n!}{(n-m)!}$
- ${}_nC_m$ "Combinations of n events taken m at a time" ${}_nC_m = \frac{n!}{m! \cdot (n-m)!}$
- $\binom{n}{m}$ "n choose m". Another way of writing ${}_nC_m$
- Sample With Replacement: Each sample is of the same population size
- Sample Without Replacement: Each time you sample, the remaining population becomes one smaller

Tree Analysis

One of determining all possible outcomes is to draw what looks like a branching tree. List all the ways you can flip a coin (Heads = 1, tails = 0)

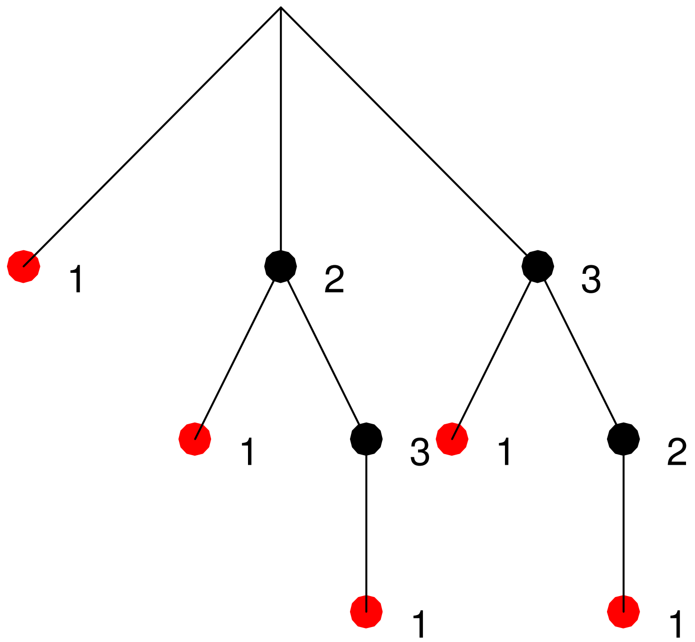
- The fist flip has two outcomes (0,1)
- The second flip has two possible outcomes
- { 111, 110, 101, 100, 011, 010, 001, 000 }



Sample Without Replacement

Given 3 keys

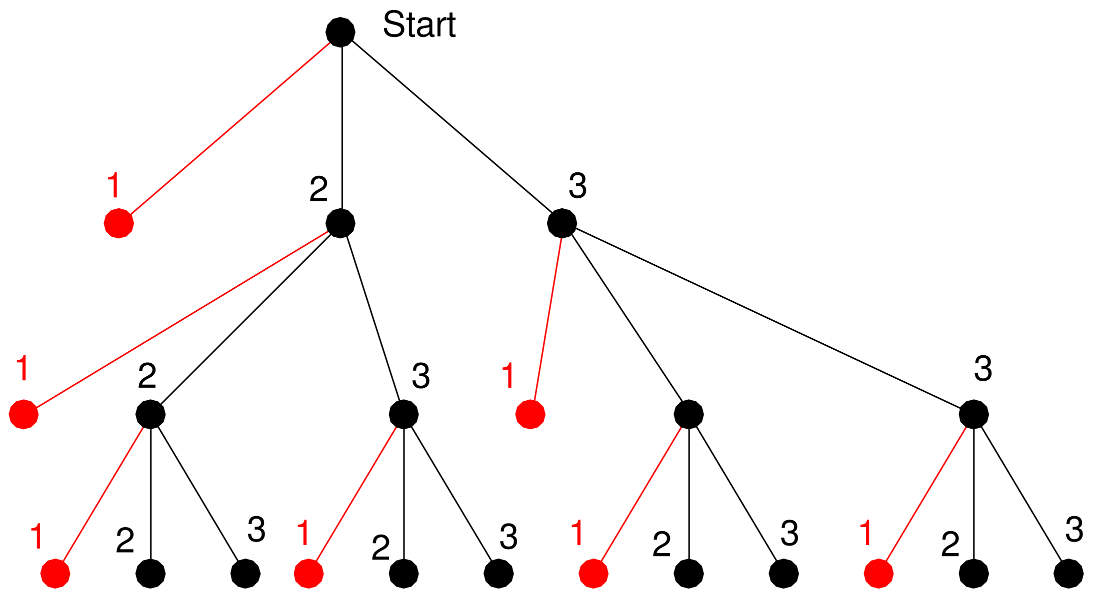
- How many tries does it take to find the right key (key #1)
- { 1, 21, 231, 31, 321 }



Sample With Replacement

Given 3 keys

- How many tries does it take to find the right key (key #1)
- { 1, 21, 221, 2221, 22221, 222221, 22222221, etc }
- Tree analysis doesn't work here (infinite combination)



Enumeration.

Assume all possibilities have equal probability.

One way to determine the probability of an event is then to

- List out all possible outcomes
- Determine how many of these are a success

The probability of a success is then

$$p(x) = \frac{\text{\# of outcomes that are a success}}{\text{Total mnumber of possible outcomes}}$$

This listing of all possible outcomes is called *enumeration*.

Example 1: Rolling a Single Die

Determine the probability of rolling a one on a six-sided die (d6).

- Assume all numbers have equal likelihood (i.e. a fair die).

Solution: List all possible outcomes

$\{ 1, 2, 3, 4, 5, 6 \}$

List all outcomes that are a success:

$\{ 1 \}$

The probability of rolling a one is thus

$p(1) = 1/6$

Example 2: Roll two dice and take the smallest number.

- Determine the probability of getting each number from 1 to 6.

Solution: With two 6-sided dice (2d6), there are 36 possible outcomes

```
(1,1)   (1,2)   (1,3)   (1,4)   (1,5)   (1,6)
(2,1)   (2,2)   (2,3)   (2,4)   (2,5)   (2,6)
(3,1)   (3,2)   (3,3)   (3,4)   (3,5)   (3,6)
(4,1)   (4,2)   (4,3)   (4,4)   (4,5)   (4,6)
(5,1)   (5,2)   (5,3)   (5,4)   (5,5)   (5,6)
(6,1)   (6,2)   (6,3)   (6,4)   (6,5)   (6,6)
```

Count how many outcomes result in a 1, 2, 3, etc.

```
1: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)
2: (2,2), (2,3), (2,4), (2,5), (2,6), (3,2), (4,2), (5,2), (6,2)
3: (3,3), (3,4), (3,5), (3,6), (4,3), (5,3), (6,3)
4: (4,4), (4,5), (4,6), (5,4), (6,4)
5: (5,5), (5,6), (6,5)
6: (6,6)
```

The probability of getting a one through six is thus

```
1:   11/36
2:  9/36
3:  7/36
4:  5/36
5:  3/36
6:  1/36
```

Note:

- The sum must add up to one
- The probability that something happens is 100%

Risk:

When it's your turn in Risk, your armies can attack a neighboring country. When you attack

- The attacker rolls one die for each attacker, up to a maximum of 3 dice
- The defender rolls one die for each defender, up to a maximum of 2 dice.

You then sort the dice, highest to lowest for the attacker and defender.

- If the attacker's highest die is more than the defender's highest die, the defender loses an army. Otherwise the attacker loses an army (defender wins on ties).
- If the defender rolled two dice, repeat for the second highest die

Problem: What is the probability that the attacker will win if it's 1 army vs. 1 army.

Solution:

- List all possible die rolls (attacker, defender).
- Note which ones where the attacker's die is more than the defenders (show in red)

```
(1,1)   (1,2)   (1,3)   (1,4)   (1,5)   (1,6)
(2,1)   (2,2)   (2,3)   (2,4)   (2,5)   (2,6)
(3,1)   (3,2)   (3,3)   (3,4)   (3,5)   (3,6)
(4,1)   (4,2)   (4,3)   (4,4)   (4,5)   (4,6)
(5,1)   (5,2)   (5,3)   (5,4)   (5,5)   (5,6)
(6,1)   (6,2)   (6,3)   (6,4)   (6,5)   (6,6)
```

- There are 10 results where the attacker wins
- There are 36 total possible results
- The probability of the attacker winning a 1 on 1 attack is 10/36

Conditional Probability

Conditional probability is the probability of outcome A happening given that B happened

$p(A|B)$ *sum the columns*

$p(B|A)$ *sum the rows*

p(Attacker Wins)	Defender Roll (x,B)							
		1	2	3	4	5	6	p(x|A)
Attacker Roll (A,x)	1	0	0	0	0	0	0	0/6
	2	1	0	0	0	0	0	1/6
	3	1	1	0	0	0	0	2/6
	4	1	1	1	0	0	0	3/6
	5	1	1	1	1	0	0	4/6
	6	1	1	1	1	1	0	5/6
p(x|B)		5/6	4/6	3/6	2/6	1/6	0/6	

You can use conditional probabilities to compute the total probability.

Problem: Determine the probability that the attacker wins if

- You attack with two armies (roll two dice) and
- The defender has only one army (rolls one die)

The probability will be

```
p(wins) = p(wins | A=1) p(A=1)  +  p(wins | A=2) p(A=2)  +  p(wins | A=3) p(A=3) +
```

From before, the probability of each result is

Highest Die for A

x	1	2	3	4	5	6
p(x)	1/36	3/36	5/36	7/36	9/36	11/36

The probability is then

	Highest Die for A					
A	1	2	3	4	5	6
p(A)	1/36	3/36	5/36	7/36	9/36	11/36
p(win|A)	0/6	1/6	2/6	3/6	4/6	5/6
p(win|A)p(A)	1/216	3/216	10/216	21/216	36/216	55/216

Adding up the bottom row you get the probability that the attacker wins

- p(attacker wins) = 126 / 216 = 0.5833

Enumeration in Matlab

As a final example, suppose

- The attacker has 3 armies (rolls 3 dice) and
- The defender has 2 armies (rolls 2 dice)

Determine the probability that

- The attacker loses 2 armies
- Both the attacker and defender lose 1 army, and
- The defender loses 2 armies.

With 5 dice being rolled, there are 7776 possible results (6^5): a few too many to enumerate by hand.

This isn't a problem for Matlab

```
% Risk   3 attacking and 2 defending dice

WINS = zeros(1,3);

for a1=1:6
    for a2=1:6
        for a3=1:6
            for d1=1:6
                for d2 = 1:6
                    N = 0;
                    A = [a1,a2,a3];
                    D = [d1,d2];
                    A = sort(A, 'descend');
                    D = sort(D, 'descend');
                    if(A(1) > D(1))
                        N = N + 1;
                    end
                    if(A(2) > D(2))
                        N = N + 1;
                    end
                    disp([A,D,N])
                    WINS(N+1) = WINS(N+1) + 1;
                end
            end
        end
    end
end
```

Debugging:

- The disp() command displays the results and lets you check if it's working correctly.
- A sample is

```
[  sorted A   ]       [ B rolls ]       A wins
 3     4     6           2     3           2     6 beats 3   4 beats 2
 3     4     6           2     4           2     6 beats 4   4 beats 2
 3     4     6           2     5           2     6 beats 5   4 beats 2
 3     4     6           2     6           1     6 loses 6   4 beats 2
```

Run this 100,000 times and you get a close approximation to the probability that

- A wins 2 times
- A wins 1 time, and
- A wins 0 times

Enumeration: Farkle Odds

Determine the probability of rolling a 3 of a kind with 6 dice

Solution:

- Run through every permutation of rolling 6 dice
- Count the number of times you get 3 of a kind

Code:

Every combination of 6 dice

- Lines 9..15

Determine the frequency of each number

- Lines 17..24
- Same as previous code

Determine if it was 3 of a kind

- Lines 25..28
- Same as previous code

```
9   for d1 = 1:6
10      for d2 = 1:6
11          for d3 = 1:6
12              for d4 = 1:6
13                  for d5 = 1:6
14                      for d6 = 1:6
15                          Dice = [d1,d2,d3,d4,d5,d6];
16                          % check for pairs
17                          N = zeros(1,6);
18                          for i=1:6
19                              for j=1:6
20                                  if(Dice(j) == i)
21                                      N(i) = N(i) + 1;
22                                  end
23                              end
24                          end
25                          [N,b] = sort(N, 'descend');
26                          if ( (N(1) == 3) & (N(4) < 3))
27                              Pair3 = Pair3 + 1;
28                          end
29                      end
30                  end
31              end
32          end
33      end
34  end
```

Results

- There are 14,700 ways to roll 3 of a kind
- There are 46,656 ways to roll 6 dice
- The probability is 0.3151

Note: This is not a random process

- There are always 14,700 ways to roll 3 of a kind
- Enumeration is exact (vs. Monte Carlo)
- Enumeration is also faster

Command Window

```
3 of a kind odds

ans =

    0.3151

Elapsed time is 1.366134 seconds.
3 of a kind odds

ans =

    0.3151

Elapsed time is 1.349371 seconds.
3 of a kind odds

ans =

    0.3151

Elapsed time is 1.351213 seconds.
fx >>
```

Enumeration: Poker odds

Determine the probability of drawing 3 of a kind using enumeration

- Go through every possible poker hand
- Count how many result in 3 of a kind

Code:

Deal every possible poker hand

- Lines 7..13

Determine the frequency of each card (A..K)

- Lines 17..20

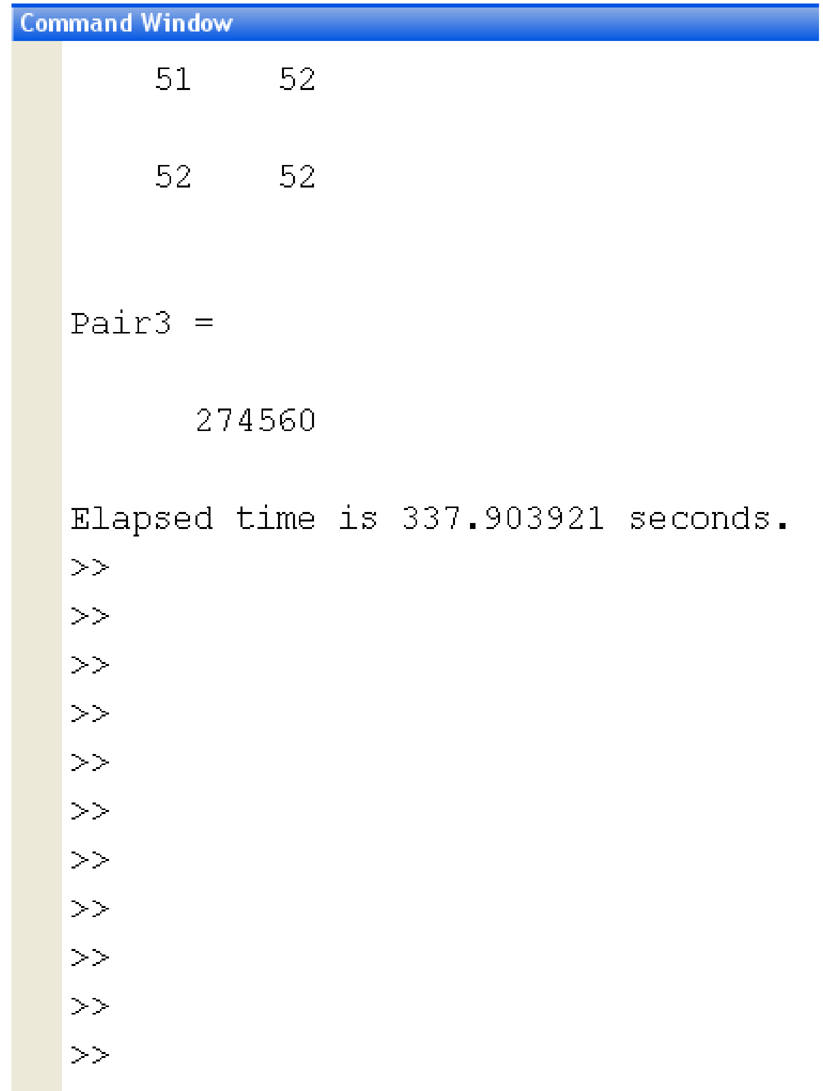
Check if you have a 3 of a kind

- Lines 22-23

```
4    tic
5    Pair3 = 0;
6
7    for c1=1:52
8      for c2 = c1:52
9          disp([c1,c2])
10       for c3 = c2:52
11         for c4 = c3:52
12           for c5 = c4:52
13             Hand = [c1,c2,c3,c4,c5];
14             Value = mod(Hand,13) + 1;
15             Suit = floor(Hand/13) + 1;
16
17             N = zeros(1,13);
18             for n=1:13
19               N(n) = sum(Value == n);
20             end
21
22             [N,a] = sort(N, 'descend');
23             if ((N(1) == 3)*(N(2) < 2)) Pair3 = Pair3
24           end
25         end
26       end
27     end
28   end
29
30   [Pair3]
```

Result:

- There are 274,560 different 3-of-a-kind poker hands
- It takes 337 seconds to go through every permutation
- The result is exact (it's not a random process)



Summary:

If

- All outcomes are equally likely, and
- You can specify each possible outcome

enumeration is one way to determine probabilities.

It can take a *long* time to go through each permutation, however.

There has to be a better tool

- There is - tomorrow's lecture.