

ECE 341 - Homework #3

Dice Games and z-Transform

Dice Games (8 Dice)

1) Assume you roll eight 6-sided dice (8d6). Using combinatorics, determine the odds of rolling 5-of-a kind:

- Roll = (xxxxx aaa), (xxxxx aa b), or (xxxxx abc)

From enumeration, $M = 1193$

Using combinatorics

$M_1 = (6c1 \text{ for } x)(8c5 \text{ for spots for } x) * (5c1 \text{ for } a)(3c3 \text{ for spots for } a)$

$$M_1 = \binom{6}{1} \binom{8}{5} * \binom{5}{1} \binom{3}{3}$$

$$M_1 = 1,680$$

$M_2 = (6c1 \text{ for } x)(8c5 \text{ for spots}) * (5c1 \text{ for } a)(3c2 \text{ for spots}) * (4c1 \text{ for } b)(1c1 \text{ for spot})$

$$M_2 = \binom{6}{1} \binom{8}{5} * \binom{5}{1} \binom{3}{2} * \binom{4}{1} \binom{1}{1}$$

$$M_2 = 20,160$$

$M_3 = (6c1 \text{ for } x)(8c5 \text{ for spots}) * (5c3 \text{ for abc})(3c1 \text{ for spot for } a)(2c1 \text{ for spot for } b)(1c1 \text{ for spot for } c)$

$$M_3 = \binom{6}{1} \binom{8}{5} * \binom{5}{3} \binom{3}{1} \binom{2}{1} \binom{1}{1}$$

$$M_3 = 20,160$$

The total is

$$M = M_1 + M_2 + M_3$$

$$M = 42,000$$

This matches enumeration results. The number of ways you can roll 8d6

$$N = 6^8 = 1,679,616$$

The odds are thus

$$p = \left(\frac{M}{N}\right) = 0.025006$$

or 2,500 in 100,000 die rolls. This matches Monte-Carlo simulation results.

2) Using combinatorics, determine the odds of rolling two tripples with 8d6:

- Roll = (xxx yyy ab) or (xxx yyy aa)

Note: your answers should match up with homework set #1.

From HW1, $M = 134,400$

$M_1 = (6c2 \text{ for } xy)(8c3 \text{ for spot for } x)(5c3 \text{ for spot for } y)*(4c1 \text{ for } a)(3c1 \text{ for } b)$

$$M_1 = \binom{6}{2} \binom{8}{3} \binom{5}{3} * \binom{4}{1} \binom{3}{1}$$

$$M_1 = 100,800$$

$M_2 = (6c2 \text{ fore } xy)(8c3 \text{ for spot for } x)*5c3 \text{ for spot for } y)(4c1 \text{ for } a)(2c2 \text{ for spot for } a)$

$$M_2 = \binom{6}{2} \binom{8}{3} \binom{5}{3} * \binom{4}{1} \binom{2}{2}$$

$$M_2 = 33,600$$

The number of ways to get two tripples is

$$M = M_1 + M_2$$

$$M = 134,400$$

This matches results from enumeration

The probability is then

$$N = 6^8 = 1,679,616$$

$$p = \left(\frac{M}{N}\right) = 0.080018$$

or 8001.83 in 100,000 rolls

- This matches Monte-Carlo results from homework #1

z-Transforms

Find the inverse z-transform

$$4) \quad X = \left(\frac{0.001z(z+1)}{(z-0.99)(z-0.96)(z-0.95)} \right)$$

Pull out a z

$$X = \left(\frac{0.001(z+1)}{(z-0.99)(z-0.96)(z-0.95)} \right) z$$

Do a partial fraction expansion

$$X = \left(\left(\frac{a}{z-0.99} \right) + \left(\frac{b}{z-0.96} \right) + \left(\frac{c}{z-0.95} \right) \right) z$$

```
>> z = 0.99;  
>> a = 0.001 * (z+1) / ((z-0.96) * (z-0.95))  
  
a = 1.6583  
  
>> z = 0.96;  
>> b = 0.001 * (z+1) / ((z-0.99) * (z-0.95))  
  
b = -6.5333  
  
>> z = 0.95;  
>> c = 0.001 * (z+1) / ((z-0.99) * (z-0.96))  
  
c = 4.8750
```

$$X = \left(\left(\frac{1.6583}{z-0.99} \right) + \left(\frac{-6.5333}{z-0.96} \right) + \left(\frac{4.8750}{z-0.95} \right) \right) z$$

Multiply through by z

$$X = \left(\frac{1.6583z}{z-0.99} \right) + \left(\frac{-6.5333z}{z-0.96} \right) + \left(\frac{4.8750z}{z-0.95} \right)$$

Take the inverse z-transform

$$x(k) = \left(1.6583(0.99)^k - 6.5333(0.96)^k + 4.8750(0.95)^k \right) u(k)$$

5) Take the inverse z-transform

$$X = \left(\frac{0.001z(z+1)}{(z-1)(z-0.99)(z-0.96)} \right)$$

Pull out a z

$$X = \left(\frac{0.001(z+1)}{(z-1)(z-0.99)(z-0.96)} \right) z$$

Do a partial fraction expansion

$$X = \left(\left(\frac{a}{z-1} \right) + \left(\frac{b}{z-0.99} \right) + \left(\frac{c}{z-0.96} \right) \right) z$$

```
>> z = 1;  
>> a = 0.001*(z+1) / ((z-0.99)*(z-0.96))  
  
a =      5.0000  
  
>> z = 0.99;  
>> b = 0.001*(z+1) / ((z-1)*(z-0.96))  
  
b =     -6.6333  
  
>> z = 0.96;  
>> c = 0.001*(z+1) / ((z-1)*(z-0.99))  
  
c =      1.6333
```

$$X = \left(\left(\frac{5}{z-1} \right) + \left(\frac{-6.6333}{z-0.99} \right) + \left(\frac{1.6333}{z-0.96} \right) \right) z$$

Multiply through by z

$$X = \left(\frac{5z}{z-1} \right) + \left(\frac{-6.6333z}{z-0.99} \right) + \left(\frac{1.6333z}{z-0.96} \right)$$

Take the inverse z-transform

$$x(k) = \left(5 - 6.3333(0.99)^k + 1.6333(0.96)^k \right) u(k)$$

6) A new Volkswagen ID.4 costs \$49,932 from Cars.com. If you take out a 60-month loan at 7% interest, what is your monthly payment? Solve using z-transforms.

Your loan balance at month k is

$$x(k+1) = 1.005833 \cdot x(k) + L \cdot \delta(k) - P \cdot u(k-1)$$

Take the z-transform

$$zX = 1.005833X + L - P\left(\frac{1}{z-1}\right)$$

$$(z - 1.005833)X = L - P\left(\frac{1}{z-1}\right)$$

$$X = \left(\frac{1}{z-1.005833}\right)L - P\left(\frac{1}{z-1}\right)\left(\frac{1}{z-1.005833}\right)$$

Do a partial fraction expansion

$$X = \left(\frac{1}{z-1.005833}\right)L - P\left(\left(\frac{-171.42}{z-1}\right) + \left(\frac{171.42}{z-1.005833}\right)\right)$$

Take the invers z-transform

$$zX = \left(\frac{z}{z-1.005833}\right)L - P\left(\left(\frac{-171.42z}{z-1}\right) + \left(\frac{171.42z}{z-1.005833}\right)\right)$$

$$zx(k) = \left((1.005833)^k L + 171.42P\left(1 - (1.005833)^k\right)\right)u(k)$$

$$x(k) = \left((1.005833)^{k-1} L + 171.42P\left(1 - (1.005833)^{k-1}\right)\right)u(k-1)$$

At k=61, the loan is zero (car is paid off after 60 payments)

$$x(61) = 0 = (1.005833)^{60}L + 171.42P\left(1 - (1.005833)^{60}\right)$$

$$P = \left(\frac{(1.005833)^{60}}{171.42P\left((1.005833)^{60}-1\right)}\right)L$$

$$P = \$988.76$$

Total payments amount to \$59,326 over 60 months.