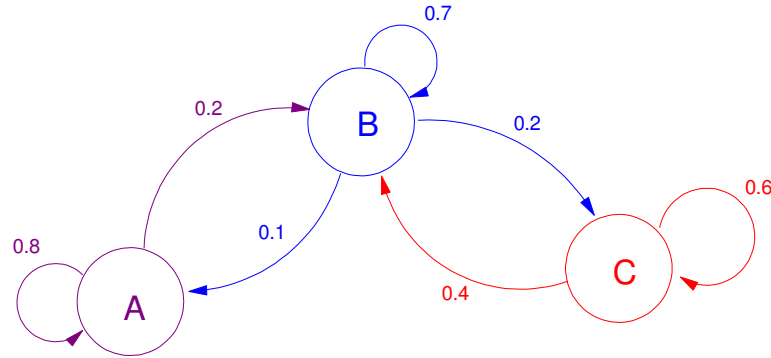


ECE 341 - Homework #11

Markov Chains.

Markov Chains

Three people are playing a game of hot potato. Each turn a player can keep the potato or pass it to another player. The probability of each decision are as follows:



1) Assume player A starts with the potato. Determine the probability that each player has the potato after 10 tosses using matrix multiplication.

Set up a state-transition matrix

$$z \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 & 0 \\ 0.2 & 0.7 & 0.4 \\ 0 & 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$zX = AX$$

At 10 tosses

$$X(10) = A^{10}X(0)$$

In Matlab

```
>> A = [0.8, 0.1, 0; 0.2, 0.7, 0.4; 0, 0.2, 0.6]
```

```
0.8000    0.1000    0
0.2000    0.7000    0.4000
0         0.2000    0.6000
```

```
>> X0 = [1; 0; 0]
```

```
1
0
0
```

```
>> A^10 * X0
```

```
0.2933
0.4810
0.2257
```

After 10 tosses, A will have the ball 29.33% of the time

2) Assume player A starts with the potato. Determine the probability that player A has the potato after k tosses using z-transforms.

- What is the probability that player A has the potato after infinite tosses?

```
>> A = [0.8, 0.1, 0; 0.2, 0.7, 0.4; 0, 0.2, 0.6]
```

```
    0.8000    0.1000    0
    0.2000    0.7000    0.4000
         0    0.2000    0.6000
```

```
>> X0 = [1; 0; 0]
```

```
    1
    0
    0
```

```
>> C = [1, 0, 0]
```

```
    1    0    0
```

```
>> D = 0;
```

```
>> G = ss(A, X0, C, D, 1);
```

```
>> zpk(G)
```

```

(z-0.9372) (z-0.3628)
-----
(z-1) (z-0.7562) (z-0.3438)
```

```
Sampling time (seconds): 1
```

The z-Transform for Y is z times this

$$\frac{(z-0.9372)(z-0.3628)z}{(z-1)(z-0.7562)(z-0.3438)}$$

Pull out a z and take a partial fraction expansion

$$A(z) = \left(\frac{(z-0.9372)(z-0.3628)}{(z-1)(z-0.7562)(z-0.3438)} \right) z$$

$$A(z) = \left(\left(\frac{a}{z-1} \right) + \left(\frac{b}{z-0.7562} \right) + \left(\frac{c}{z-0.3438} \right) \right) z$$

Doing a partial fraction expansion in Matlab

```
>> P = eig(A)
```

```
    1.0000
    0.7562
    0.3438
```

```
>> z = P(1) + 1e-6;
```

```
>> a = evalfr(G, z) * (z-P(1))
```

```
a =    0.2500
```

```
>> z = P(2) + 1e-6;
```

```
>> b = evalfr(G, z) * (z-P(2))
```

```
b =    0.7085
```

```
>> z = P(3) + 1e-6;
>> c = evalfr(G,z) * (z-P(3))
c = 0.0415
```

so

$$A(z) = \left(\left(\frac{0.2500}{z-1} \right) + \left(\frac{0.7085}{z-0.7562} \right) + \left(\frac{0.0415}{z-0.3438} \right) \right) z$$

Multiply through by z and take the inverse z-transform

$$a(k) = \left(0.2500(1)^k + 0.7085(0.7562)^k + 0.0415(0.3438)^k \right) u(k)$$

At k=10

```
>> k = 10;
>> a = 0.25 + 0.7085*(0.7562^k) + 0.0415*(0.3438^k)

a = 0.2933
```

This matches problem #1

3) Assume player A starts with the potato. Determine the probability that player A has the potato after k tosses using eigenvalues and eigenvectors.

In general, the response is

$$x(t) = a_1 \Lambda_1 e^{\lambda_1 t} + a_2 \Lambda_2 e^{\lambda_2 t} + a_3 \Lambda_3 e^{\lambda_3 t}$$

where

- Λ_i is the i th eigenvector and
- λ_i is the i th eigenvalue.

Using Matlab, the eigenvalues and eigenvectors can be found

```
>> A = [0.8, 0.1, 0; 0.2, 0.7, 0.4; 0, 0.2, 0.6]
```

```
    0.8000    0.1000         0
    0.2000    0.7000    0.4000
         0    0.2000    0.6000
```

```
>> [M, V] = eig(A)
```

```
-0.4082    0.8144    0.1703
-0.8165   -0.3571   -0.7767
-0.4082   -0.4574    0.6064

 1.0000         0         0
         0    0.7562         0
         0         0    0.3438
```

The initial conditions tell you the weightings on a_i

$$x(0) = a_1 \Lambda_1 + a_2 \Lambda_2 + a_3 \Lambda_3$$

$$x(0) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Finding these...

```
>> X0 = [1; 0; 0]
```

```
    1
    0
    0
```

```
>> A = inv(M) * X0
```

```
a1    -0.6124    initial condition on the 1st eigenvector
a2     0.8699    initial condition on the 2nd eigenvector
a3     0.2438    initial condition on the 3rd eigenvector
```

Note that a_i is a scalar that can be combined with eigenvector i . This results in the total response being

```
>> A(1)*M(:,1)
```

```
0.2500  
0.5000  
0.2500
```

```
>> A(2)*M(:,2)
```

```
0.7085  
-0.3106  
-0.3979
```

```
>> A(3)*M(:,3)
```

```
0.0415  
-0.1894  
0.1479
```

This tells you that

$$X(k) = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.25 \end{bmatrix} (1)^k + \begin{bmatrix} 0.7085 \\ -0.3106 \\ -0.3979 \end{bmatrix} (0.7562)^k + \begin{bmatrix} 0.0415 \\ -0.1894 \\ 0.1479 \end{bmatrix} (0.3438)^k$$

At $k = 10$

$$X(10) = \begin{bmatrix} 0.2933 \\ 0.4810 \\ 0.2257 \end{bmatrix}$$

which matches problem #1

Markov Chains with Absorbing States

Problem 4 & 5: Two teams, A and B, are playing a match made up of N games. For each game

- Team A has a 50% chance of winning
- There is a 20% chance of a tie, and
- Team B has a 30% chance of winning

In order to win the match, a team must be up by 2 games.

4) Determine the probability that team A wins the match after k games for $k = \{0 \dots 10\}$ using matrix multiplication.

Start defining the states

$$X(k) = \begin{bmatrix} \text{A Wins (up 2)} \\ \text{Up 1 game} \\ \text{Even} \\ \text{Down 1} \\ \text{B Wins (down 2)} \end{bmatrix}$$

The state transition matrix is then

$$X(k+1) = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0.2 & 0.5 & 0 \\ 0 & 0 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 & 1 \end{bmatrix} X(k)$$

$$X(k+1) = A \cdot X(k)$$

At 10 games

$$X(10) = A^{10}X(0)$$

In Matlab

```
>> A = [1,0.5,0,0,0;0,0.2,0.5,0,0;0,0.3,0.2,0.5,0;0,0,0.3,0.2,0;0,0,0,0.3,1]
```

```

1.0000    0.5000         0         0         0
         0    0.2000    0.5000         0         0
         0    0.3000    0.2000    0.5000         0
         0         0    0.3000    0.2000         0
         0         0         0    0.3000    1.0000
```

```
>> X0 = [0;0;1;0;0]
```

```

0
0
1
0
0
```

```
>> A^10 * X0
```

```
0.6859  
0.0249  
0.0273  
0.0150  
0.2469
```

After 10 games, the status of the match is...

68.59%	A has won the match
02.49%	match is ongoing with A up one game
02.73%	match is ongoing and tied
01.50%	match is ongoing and B is up one game
24.69%	B has won the match

5) Determine the z-transform for the probability that A wins the match after k games

- From the z transforms, determine the explicit function for p(A) wins after game k.

First, find the z-transform for A(z)

```
>> A = [1,0.5,0,0,0;0,0.2,0.5,0,0;0,0.3,0.2,0.5,0;0,0,0.3,0.2,0;0,0,0,0.3,1]
```

```
1.0000    0.5000         0         0         0
         0    0.2000    0.5000         0         0
         0    0.3000    0.2000    0.5000         0
         0         0    0.3000    0.2000         0
         0         0         0    0.3000    1.0000
```

```
>> X0 = [0;0;1;0;0]
```

```
0
0
1
0
0
```

```
>> C = [1,0,0,0,0]
```

```
1         0         0         0         0
```

```
>> D = 0;
```

```
>> G = ss(A, X0, C, D, 1);
```

```
>> zpkk(G)
```

```

          0.25 (z-0.2)
-----
(z-1) (z-0.7477) (z-0.2) (z+0.3477)
```

Multiply by z to get A(z)

$$A(z) = \left(\frac{0.25(z-0.2)z}{(z-1)(z-0.7477)(z-0.2)(z+0.3477)} \right)$$

Take the inverse z-transform. Pull out a z

$$A(z) = \left(\frac{0.25(z-0.2)}{(z-1)(z-0.7477)(z-0.2)(z+0.3477)} \right) z$$

Do a partial fraction expansion

$$A(z) = \left(\left(\frac{a}{z-1} \right) + \left(\frac{b}{z-0.7477} \right) + \left(\frac{c}{z-0.2} \right) + \left(\frac{d}{z-0.3477} \right) \right) z$$

In Matlab

```
>> P = eig(A)
```

```
1.0000
1.0000
-0.3477
0.7477
0.2000
```

```

>> z = P(1) + 1e-6;
>> a = evalfr(G,z) * (z-1)

a =    0.7353

>> z = P(4) + 1e-6;
>> b = evalfr(G,z) * (z-P(4))

b =   -0.9046

>> z = P(5) + 1e-6;
>> c = evalfr(G,z) * (z-P(5))

c =  1.0417e-006

>> z = P(3) + 1e-6;
>> d = evalfr(G,z) * (z-P(3))

d =    0.1693

>>

```

or

$$A(z) = \left(\left(\frac{0.7353}{z-1} \right) + \left(\frac{-0.9046}{z-0.7477} \right) + \left(\frac{0}{z-0.2} \right) + \left(\frac{0.1693}{z-0.3477} \right) \right) z$$

Multiply through by z and take the inverse z transform

$$a(k) = \left(0.7353(1)^k - 0.9046(0.7477)^k + 0.1693(0.3477)^k \right) u(k)$$

Plug in k=10

```

>> k = 10;
>> a = 0.7353 - 0.9046*(0.7477)^k + 0.1693*(0.3477)^k

a =    0.6859

```

which matches problem #4