

ECE 341 - Homework #6

LaPlace Transforms, Continuous Probability Density Functions

LaPlace Transforms

Let X and Y be related by the following transfer function:

$$Y = \left(\frac{k(s+2)}{(s+3)(s+7)} \right) X$$

1a) Find k so this is a valid pdf

Translation: The zeroth-moment must be 1.000

$$\left(\frac{k(s+2)}{(s+3)(s+7)} \right)_{s=0} = 1$$

$$k = 10.5$$

1b) Find the differential equation which relates X and Y

Multiply out the polynomials

$$Y = \left(\frac{10.5(s+2)}{s^2+10s+21} \right) X$$

Cross multiply

$$(s^2 + 10s + 21)Y = 10.5(s + 2)X$$

'sY' means 'the derivative of y'

$$y'' + 10y' + 21y = 10.5x' + 21x$$

2) Determine $y(t)$ if $x(t)$ is an impulse (i.e. find the pdf for $y(t)$)

$$Y = \left(\frac{10.5(s+2)}{(s+3)(s+7)} \right) X$$

$$Y = \left(\frac{10.5(s+2)}{(s+3)(s+7)} \right) \cdot 1$$

Do a partial fraction expansion

$$Y = \left(\frac{-2.6250}{s+3} \right) + \left(\frac{13.1250}{s+7} \right)$$

Take the inverse LaPlace transform

$$y(t) = (-2.6250 \cdot e^{-3t} + 13.2350 \cdot e^{-7t})u(t)$$

This is the pdf for $y()$

3) Determine $y(t)$ if $x(t)$ is a unit step: $1/s$ (i.e. find the cdf for $y(t)$)

$$Y = \left(\frac{10.5(s+2)}{(s+3)(s+7)} \right) X$$

$$Y = \left(\frac{10.5(s+2)}{(s+3)(s+7)} \right) \left(\frac{1}{s} \right)$$

Do a partial fraction expansion

$$Y = \left(\frac{0.8750}{s+3} \right) + \left(\frac{-1.8750}{s+7} \right) + \left(\frac{1}{s} \right)$$

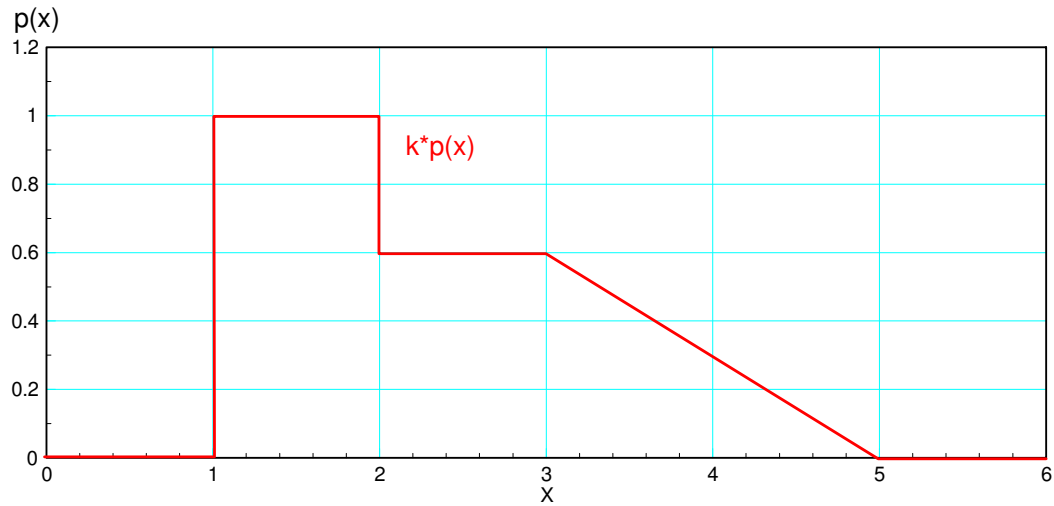
Take the inverse LaPlace transform

$$y(t) = (0.8750 \cdot e^{-3t} - 1.8750 \cdot e^{-7t} + 1)u(t)$$

This is the cdf for Y

Continuous Probability Density Functions

pdf for A shown



4) Scale so this is a valid pdf (area = 1)

As drawn the total area is 2.2

$$k = \frac{1}{2.2} = 0.4545$$

5) Find the cdf for A

$$2.2 \cdot pdf = \begin{cases} 0 & x < 1 \\ 1 & 1 < x < 2 \\ 0.6 & 2 < x < 3 \\ 1.5 - 0.3x & 3 < x < 5 \\ 0 & x > 5 \end{cases}$$

Integrate

- Add integration constants to make the cdf continuous

$$2.2 \cdot cdf = \begin{cases} 0 & x < 1 \\ x - 1 & 1 < x < 2 \\ 0.6x - 0.2 & 2 < x < 3 \\ 1.5x - 0.15x^2 - 1.55 & 3 < x < 5 \\ 2.2 & x > 5 \end{cases}$$

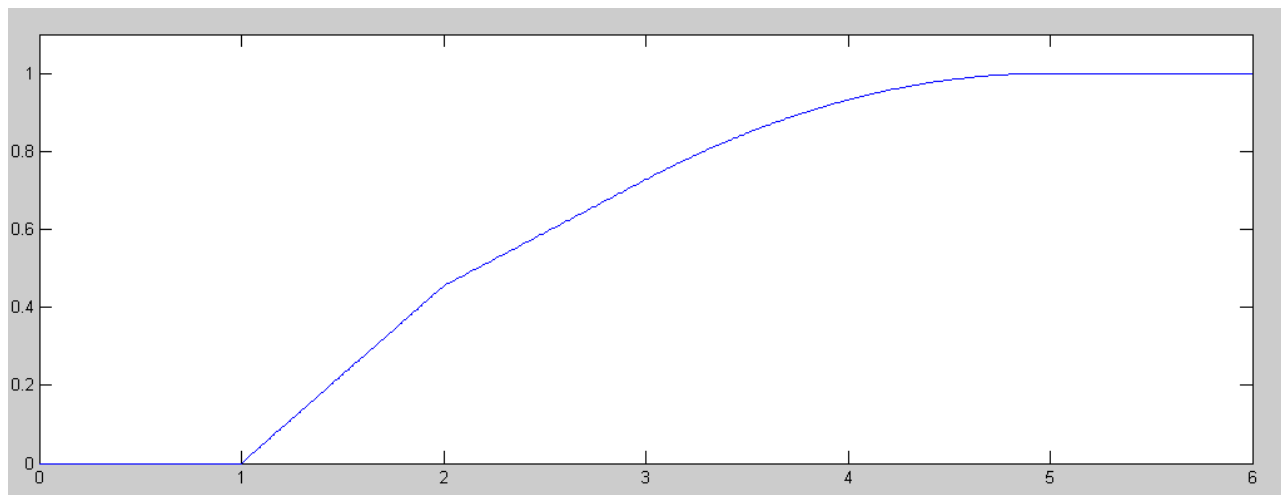
6) Determine 20 random values with a pdf of A

Start with a Matlab function for the cdf

```
function [y] = cdf(x)
if(x < 1)
    y = 0;
elseif(x<2)
    y = x-1;
elseif(x<3)
    y = 0.6*x - 0.2;
elseif(x<5)
    y = 1.5*x - 0.15*x^2 - 1.55;
else
    y = 2.2;
end
y = y / 2.2;
end
```

Check that it's a valid cdf

```
>> x = [0:0.01:10]';
>> y = 0*x;
>> for i=1:length(x)
    y(i) = cdf(x(i));
end
>> plot(x,y)
>> ylim([0,1.1])
```



This is a function where you pass x and it returns y. Create a function that goes the other way

- Use California method to find $f(x) = 0$

```

function [x] = inverse_cdf(p)

x1 = 0;
y1 = cdf(x1) - p;

x2 = 10;
y2 = cdf(x2) - p;

for i=1:10
    x3 = x2 - (x2-x1)/(y2-y1)*y2;
    y3 = cdf(x3) - p;
    if(y3<0)
        x1 = x3;
        y1 = y3;
    else
        x2 = x3;
        y2 = y3;
    end
end
x = x3;
end

```

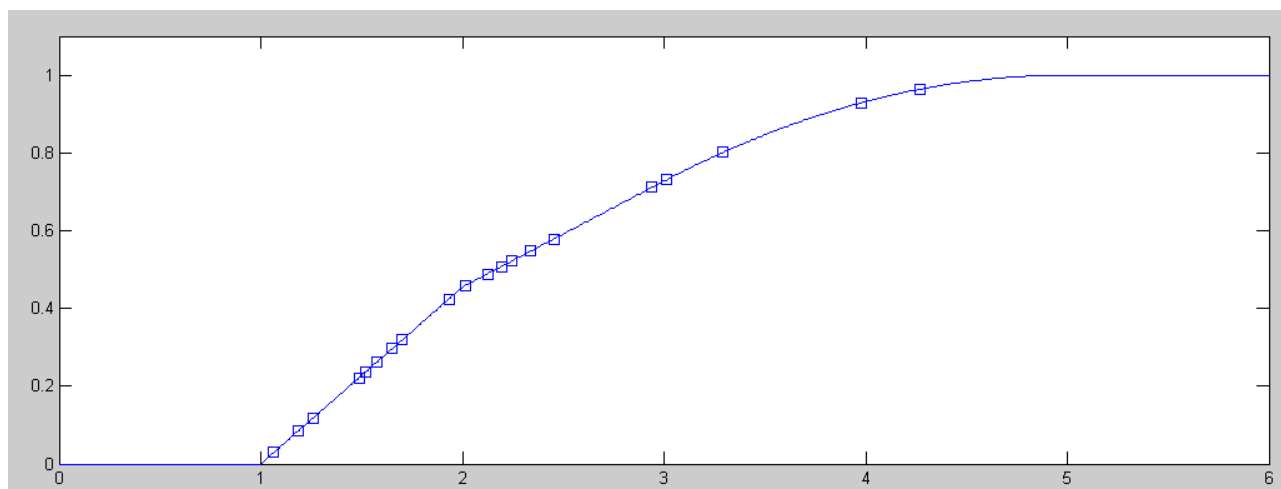
Now find 20 random values for A

```

for i=1:20
    p = rand;
    x = inverse_cdf(p);
    disp([i,p,x])
end

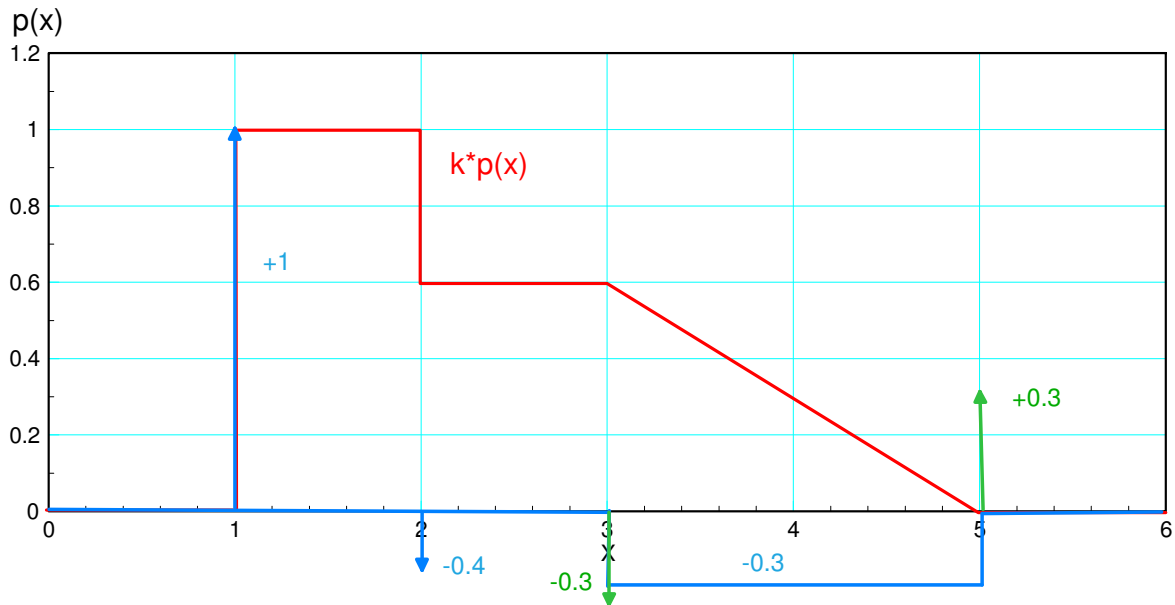
```

i	p	x
1.0000	0.7112	2.9411
2.0000	0.2217	1.4878
3.0000	0.1174	1.2583
4.0000	0.2967	1.6527
5.0000	0.3188	1.7013
6.0000	0.4242	1.9332
7.0000	0.5079	2.1955
8.0000	0.0855	1.1881
9.0000	0.2625	1.5775
10.0000	0.8010	3.2917
11.0000	0.0292	1.0643
12.0000	0.9289	3.9789
13.0000	0.7303	3.0112
14.0000	0.4886	2.1249
15.0000	0.5785	2.4546
16.0000	0.2373	1.5220
17.0000	0.4588	2.0158
18.0000	0.9631	4.2696
19.0000	0.5468	2.3383
20.0000	0.5211	2.2442



7) Find the Laplace transform of A

- Take the derivative (shown in blue)
- Take the second derivative (shown in green)



Derivative (blue)

$$\left(\frac{1}{s}\right)(1e^{-s} - 0.4e^{-2s})$$

Second derivative (green)

$$\left(\frac{1}{s^2}\right)(-0.3e^{-3s} + 0.3e^{-5s})$$

Put it together

$$2.2Y(s) = \left(\frac{1}{s}\right)(1e^{-s} - 0.4e^{-2s}) + \left(\frac{1}{s^2}\right)(-0.3e^{-3s} + 0.3e^{-5s})$$