ECE 341 - Homework #11

Markov Chains.

Problem 1 & 2) Two teams, A and B, are playing a match made up of N games. For each game

- Team A has a 45% chance of winning
- There is a 15% chance of a tie, and
- Team B has a 40% chance of winning

In order to win the match, a team must be **up** by 2 games.

1) Determine the probabilty that team A wins the match after k games for k = {0 ... 10} using matrix multiplication.

The state-transition matrix is

$$\begin{bmatrix} A_2 \\ A_1 \\ A_0 \\ A_{-1} \\ A_{-2} \end{bmatrix} = \begin{bmatrix} 1 & 0.45 & 0 & 0 & 0 \\ 0 & 0.15 & 0.45 & 0 & 0 \\ 0 & 0.4 & 0.15 & 0.45 & 0 \\ 0 & 0 & 0.4 & 0.15 & 0 \\ 0 & 0 & 0 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ A_1 \\ A_0 \\ A_{-1} \\ A_{-2} \end{bmatrix}$$

In Matlab

```
>> A=[1,0.45,0,0,0;0,0.15,0.45,0,0;0,0.4,0.15,0.45,0;0,0,0.4,0.15,0;0,0,0,0.4,1]

    1.0000    0.4500         0         0         0
         0    0.1500    0.4500         0         0
         0    0.4000    0.1500    0.4500         0
         0         0    0.4000    0.1500         0
         0         0         0    0.4000    1.0000

>> X0 = [0,0,1,0,0]'

     0
     0
     1
     0
     0

>> G0 = [1,0,0,0,0] * A^0 * X0

G0 =      0

>> G1 = [1,0,0,0,0] * A^1 * X0

G1 =      0

>> G2 = [1,0,0,0,0] * A^2 * X0

G2 =      0.2025


>> G3 = [1,0,0,0,0] * A^3 * X0
```

```
G3 =   0.2633

>> G4 = [1,0,0,0,0] * A^4 * X0

G4 =   0.3498

>> G5 = [1,0,0,0,0] * A^5 * X0

G5 =     0.3963

>> G6 = [1,0,0,0,0] * A^6 * X0

G6 =     0.4395

>> G7 = [1,0,0,0,0] * A^7 * X0

G7 =     0.4681

>> G8 = [1,0,0,0,0] * A^8 * X0

G8 =     0.4912

>> G9 = [1,0,0,0,0] * A^9 * X0

G9  =    0.5079

>> G10 = [1,0,0,0,0] * A^10 * X0

G10 =    0.5206
```

2) Determine the z-transform for the probability that A wins the match after k games
   - From the z transforms, determine the explicit function for p(A) wins after game k.

Find the z-transform

```
>> X0 = [0;0;1;0;0];
>> C = [1,0,0,0,0];
>> G = ss(A, X0, C, 0, 1);
>> zpk(G)

Zero/pole/gain:
        0.2025 (z-0.15)
--------------------------------
(z-1) (z-0.75) (z+0.45) (z-0.15)

Sampling time (seconds): 1
```

Multuply by z to get the z-transform for p(k)

$$P = \left(\frac{0.2025z}{(z-1)(z-0.75)(z+0.45)} \right)$$

Taking the inverse z-trasform...

Find the partial fraction expansion

$$P = \left(\frac{0.2025}{(z-1)(z-0.75)(z+0.45)} \right) z$$

$$P = \left(\left(\frac{0.5586}{z-1} \right) + \left(\frac{-0.6750}{z-0.75} \right) + \left(\frac{0.1164}{z+0.45} \right) \right) z$$

$$P = \left(\frac{0.5586z}{z-1} \right) + \left(\frac{-0.6750z}{z-0.75} \right) + \left(\frac{0.1164z}{z+0.45} \right)$$

$$p(k) = \left(0.5586 - 0.6750(0.75)^k + 0.1164(-0.45)^k \right) u(k)$$

Solving in Matlab

```
for k=1:10
    p = 0.5586 - 0.6750*(0.75^k) + 0.1164*(-0.45)^k;
    disp([k,p])
end
```

k	p(k)	problem 1
1.0000	-0.0000	0
2.0000	0.2025	0.2025
3.0000	0.2632	0.2633
4.0000	0.3498	0.3498
5.0000	0.3963	0.3963
6.0000	0.4394	0.4395
7.0000	0.4681	0.4681
8.0000	0.4912	0.4819
9.0000	0.5078	0.5079
10.0000	0.5206	0.5206

3) Two players are playing a game of tennis. To win a game, a player must win 4 points *and* be up by 2 points.

- If player A reaches 4 points and player B has less than 3 points, the game is over and player A wins.
- If player A reaches 4 points and player B has 3 points, then the game reverts to 'win by 2' rules. Both players keep playing until one of them is up by 2 games.

Supppose:

- Player A has a 55% chance of winning any given point
- Player B has a 45% chance of winning any given point.

What is the probabilty that player A wins the game (first to 4 games, win by 2)?

- Note: This is a combination of a binomial distribution (A has 4 points while B has 0, 1, or 2 points) along with a Markov chain (A and B both have 3 points, at which point it becomes a win-by-2 series)

The ways A can win are

- a) A wins 3 of first 3 games then A wins game 4 (A up 3-0 then wins)
- b) A wins 3 of first 4 games then game 5 (A up 3-1 then A wins)
- c) A wins 3 of rist 5 games then game 6) (A up 3-2 then A wins)
- d) A and B are tied (3,3) then A wins a best-of-2 series (Markov chain)

This is a conditional probability

a) A wins fist 3 of 3 games then A wins game 4

$$p(a) = \binom{3}{3}(0.55)^3(0.45)^0 = 0.16638$$

$$p(A|a)p(a) = (0.55)(0.16638) = 0.09151$$

b) A wins 3 of first 4 games then wins game 5

$$p(b) = \binom{4}{3}(0.55^3(0.45))^1 = 0.29948$$

$$p(A|b)p(b) = (0.55)(0.29948) = 0.16471$$

c) A wins 3 of first 5 games then wins game 6

$$p(c) = \binom{5}{3}(0.55)^3(0.45)^2 = 0.33691$$

$$p(A|c)p(c) = 0.18530$$

d) A wins best-of-two series starting at tied: 3-3 (d)

$$p(d) = \binom{6}{3}(0.55)^3(0.45)^3 = 0.30322$$

p(A) winning from here comes from a Markov chain

$$z\begin{bmatrix} p2 \\ p1 \\ p0 \\ m1 \\ m2 \end{bmatrix} = \begin{bmatrix} 1 & 0.55 & 0 & 0 & 0 \\ 0 & 0 & 0.55 & 0 & 0 \\ 0 & 0.45 & 0 & 0.55 & 0 \\ 0 & 0 & 0.45 & 0 & 0 \\ 0 & 0 & 0 & 0.45 & 1 \end{bmatrix}\begin{bmatrix} p2 \\ p1 \\ p0 \\ m1 \\ m2 \end{bmatrix}$$

Finding the result after 100 matches in Matlab

```
>> A = [1,0.55,0,0,0;0,0,0.55,0,0;0,0.45,0,0.55,0;0,0,0.45,0,0;0,0,0,0.45,1]

    1.0000    0.5500         0         0         0
         0         0    0.5500         0         0
         0    0.4500         0    0.5500         0
         0         0    0.4500         0         0
         0         0         0    0.4500    1.0000

>> A^100

    1.0000    0.8196    0.5990    0.3295         0
         0    0.0000         0    0.0000         0
         0         0    0.0000         0         0
         0    0.0000         0    0.0000         0
         0    0.1804    0.4010    0.6705    1.0000

>>
```

If you start out 0-0 (column #3), A wins (row #1) 59.90% of the time

$$p(A|d) = 0.5990$$

and

$$p(A|d)p(d) = 0.18163$$

The total chances of A winning are then

$$p(A) = p(A|a)p(a) + p(A|b)p(b) + p(A|c)p(c) + p(A|d)p(d)$$

$$p(A) = 0.62315$$

With this format, player A has a 62.315% chance of winnign the match