ECE 341 - Homework #2

Card Games & z-Transforms

The card game *bridge* uses a 52-card deck. Each person is dealt 13 cards for their hand.

1) How many different hands are possible? (order doesn't matter)

$$N = \binom{52}{13} = \left(\frac{52!}{13!\cdot(52-13)!}\right) = 635,013,559,600$$

2) What is the probability of having 9 cards of one suit in your hand?

The number of hands with 9 cards of one suit are

M = (4 suits, choose 1) (13 cards in that suit, choose 9)(39 remaining cards choose 4)

$$M = \binom{4}{1}\binom{13}{9}\binom{39}{4} = 235,237,860$$

the probability is

$$p = \left(\frac{M}{N}\right) = \left(\frac{\binom{4}{1}\binom{13}{9}\binom{39}{4}}{\binom{52}{13}}\right) = 0.000370$$

or 2699.45 : 1 odds against

3) What is the probability of having no points (no Jacks, Queens, Kings, or Aces)?

16 cards have points, 36 do not

M = (36 cards with no points, choose 13) (16 cards with points, choose 0)

$$M = \binom{36}{13}\binom{16}{0} = 2,310,789,600$$

$$p = \left(\frac{M}{N}\right) = \left(\frac{\binom{36}{13}\binom{16}{0}}{\binom{52}{13}}\right) = 0.003639$$

or 274.81 : 1 odds againse

In 4-card poker, you're dealt just 4 cards

4) Compute the odds of 2-pair in 4-card poker

hand = xx yy

hands = (52 choose 4)

$$N = \binom{52}{4} = 270,725$$

hands that are 2 pair are

M = (13 values choose 2)(4 cards of 1st value, choose 2)(4 cards of 2nd value choose 2)

$$M = \binom{13}{2}\binom{4}{2}\binom{4}{2} = 2808$$

$$p = \left(\frac{M}{N}\right) = \left(\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}}{\binom{52}{4}}\right) = 0.010372$$

From enumeration (homework #1), M = 2808

5) Compute the odds being dealt one-pair

hand = xx y z

M = (13 values choose 1)(4 cards choose 2)(12 remaining values choose 2)(4 values choose 1)(4 c 1)

$$M = \binom{13}{1}\binom{4}{2}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 82,368$$

$$p = \left(\frac{\binom{13}{1}\binom{4}{2}\binom{48}{1}\binom{44}{1}}{\binom{52}{4}}\right) = 0.30425$$

From enumeration (homework #1), M = 82,368

6) Determine the odds of a 2-pair and 1-pair using Matlab and a Monte-Carlo simulation and 1 million hands of 4-card poker

Code:

```
% ECE 341 Homework #2
% 4-Card Stud

tic
Pair22 = 0;
Pair2 = 0;

for i0 = 1:1e5

   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:4);
   Value = mod(Hand,13) + 1;
   Suit = ceil(Hand/13);

   N = zeros(1,13);
   for n=1:13
       N(n) = sum(Value == n);
   end

   [N,a] = sort(N, 'descend');

   if ( (N(1)==2)*(N(2)==2) ) Pair22 = Pair22 + 1; end
   if ( (N(1)==2)*(N(2)==1) ) Pair2 = Pair2 + 1; end

end

disp([Pair22, Pair2]/1e5)
toc
```

Result

```
 2-pair         pair
0.01092        0.30437

Elapsed time is 10.824297 seconds.
```

Conditional Probability in 4-Card Poker

7) Compute the probability of getting 4-of-a-kind if there is a single draw step

• If you are dealt 4-of-a-kind, draw no cards	hand = xxxx	draw 0
• If you are dealt 3-of-a-kind, draw one card	hand = xxxy	discard y, draw 1
• If you are dealt 2-pair or 2-of-a-kind, draw 2 cards	hand = xxyz	discard yz, draw 2
• If you are dealt no pairs, draw 3 cards.	hand = xyzt	discard yzt, draw 3

4-of a kind

$$M = \binom{13}{1}\binom{4}{1} = 13$$

p(getting 4-of-a-kind) from here

p = 1

p(3 of a kind)

$$M = \binom{13}{1}\binom{4}{3}\binom{48}{1} = 2496$$

p(getting 4-of-a-kind) from here is

$$p = \tfrac{1}{47}$$

p(2 of a kind)

$$M = \binom{13}{1}\binom{4}{2}\binom{48}{2} = 87,984$$

p(getting 4-of-a-kind from here) is

$$p = \left(\frac{\binom{2}{2}}{\binom{47}{2}} \right) = 0.000925$$

p(high card)

$$M = \binom{13}{4}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 183,040$$

p(drawing 4 of a kind when keeping one card)

$$p = \left(\frac{\binom{3}{3}}{\binom{47}{3}} \right) = 0.000062$$

The total odds are then

$$p = p(4|4) \cdot p(4) + p(4|3) \cdot p(3) + p(4|2) \cdot p(2) + p(4|1) \cdot p(1)$$

$$p = 1 \cdot \left(\frac{4}{\binom{52}{4}} \right) + \left(\frac{1}{47} \right) \left(\frac{2496}{\binom{52}{4}} \right) + \left(\frac{\binom{2}{2}}{\binom{47}{2}} \right) \left(\frac{87984}{\binom{52}{4}} \right) + \left(\frac{\binom{3}{3}}{\binom{47}{3}} \right) \left(\frac{183040}{\binom{52}{4}} \right)$$

```
p = 0.000015 + 0.000196 + 0.000301 + 0.000042

p = 0.000554
```

8) Check your answers using a Monte Carlo simulation with 1 million hands of 4-card draw poker

Result

```
Pair4 =   643 out of 1 million

Elapsed time is 77.263381 seconds
```

p = 0.000 643 (Monte-Carlo)

p = 0.000 554 (computed)

Code:

```
% ECE 341 Homework #2
% 4-Card draw looking for 4 of a kind

tic
Pair4 = 0

clc

for i0 = 1:1e6

   X = rand(1,52);
   [a,Deck] = sort(X);
   Hand = Deck(1:4);
   Value = mod(Hand,13) + 1;
   Suit = ceil(Hand/13);

   N = Suit / 10;
   for i=1:4
       for j=1:4
           if(Value(i) == Value(j))
               N(i) = N(i) + 1;
           end
       end
   end

   [a,b] = sort(N, 'descend');
   Hand = Hand(b);
   Value = Value(b);
   Suit = Suit(b);

   N = N(b);
   N = floor(N);

   % draw cards

   if(N(1) == 4)  end
   if(N(1) == 3)  Hand(4) = Deck(5); end
   if(N(1) == 2)  Hand(3:4) = Deck(5:6);  end
   if(N(1) == 1)  Hand(2:4) = Deck(5:7);  end

   Value = mod(Hand,13) + 1;
   Suit = floor(Hand/13) + 1;

   N = Suit / 10;
   for i=1:4
       for j=1:4
           if(Value(i) == Value(j))
```

```
               N(i) = N(i) + 1;
           end
       end
   end
   N = floor(N);

   if ( N(1) == 4 ) Pair4 = Pair4 + 1; end

end

[Pair4]
toc
```

9) A new Tesla Model Y ciosts \$58,990. If you take out a 36-month load at 2.34% interest, what is your monthly payment? Solve using z-transforms.

Assuming you make constant payments (p) each month starting at monthe #1, the load balance next month is

$$x(k+1) = \left(1 + \tfrac{0.0234}{12}\right)x(k) + (\$58,990)\delta(k) - p \cdot u(k-1)$$

$$x(k+1) = (1+a)x(k) + L\delta(k) - Pu(k-1)$$

or using z-transforms

$$zX = (1+a)X + L - P\left(\tfrac{1}{z-1}\right)$$

Doing some algebra

$$(z-(1+a))X = L - P\left(\tfrac{1}{z-1}\right)$$

$$X = \left(\tfrac{1}{z-(1+a)}\right)L - P\left(\tfrac{1}{(z-1)(z-(1+a))}\right)$$

$$X = \left(\tfrac{1}{z-(1+a)}\right)L - P\left(\tfrac{-1/a}{z-1} + \tfrac{1/a}{z-(1+a)}\right)$$

Take the inverse z-transform

$$zX = \left(\tfrac{z}{z-(1+a)}\right)L - \left(\tfrac{P}{a}\right)\left(\tfrac{-z}{z-1} + \tfrac{z}{z-(1+a)}\right)$$

$$zx(k) = \left((1+a)^k L - \tfrac{P}{a}\left((1+a)^k - 1\right)\right)u(k)$$

$$x(k) = \left((1+a)^{k-1} L - \tfrac{P}{a}\left((1+a)^{k-1} - 1\right)\right)u(k-1)$$

After 36 payments, the balance is zero

$$x(36) = 0 = \left((1+a)^{35} L - \tfrac{P}{a}\left((1+a)^{35} - 1\right)\right)$$

Your monthly payments are thus

$$P = \left(\tfrac{a\cdot(1+a)^{35}}{(1+a)^{35}-1}\right)L$$

Plugging in numbers

$$P = \left(\tfrac{0.00195\cdot(1.00195)^{35}}{1.00195^{35}-1}\right)\cdot \$58,990$$

$$P = \$1745.24$$

Your monthly payments to own a Tesla Model Y would be \$1745.24

- \$1834.47 / month at 5.74% interest (car loan interest rate as of May 20, 2022)