Complex Numbers and Phasors

ECE 320 Electronics I

Jake Glower - Lecture #2

Complex Numbers:

Define

$$j = \sqrt{-1}$$

$$j^2 = -1$$

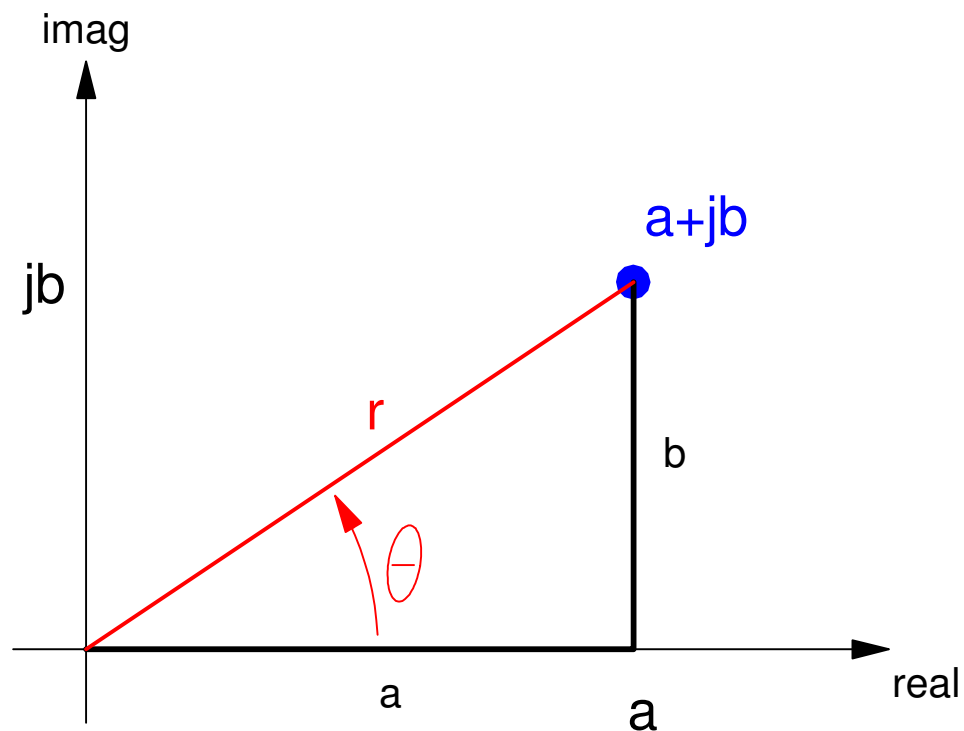$$e^{j\theta} = \cos\theta + j\sin\theta$$

Complex numbers have 2 terms:

- Rectangular Form

  $$X = a + jb$$

- Polar Form
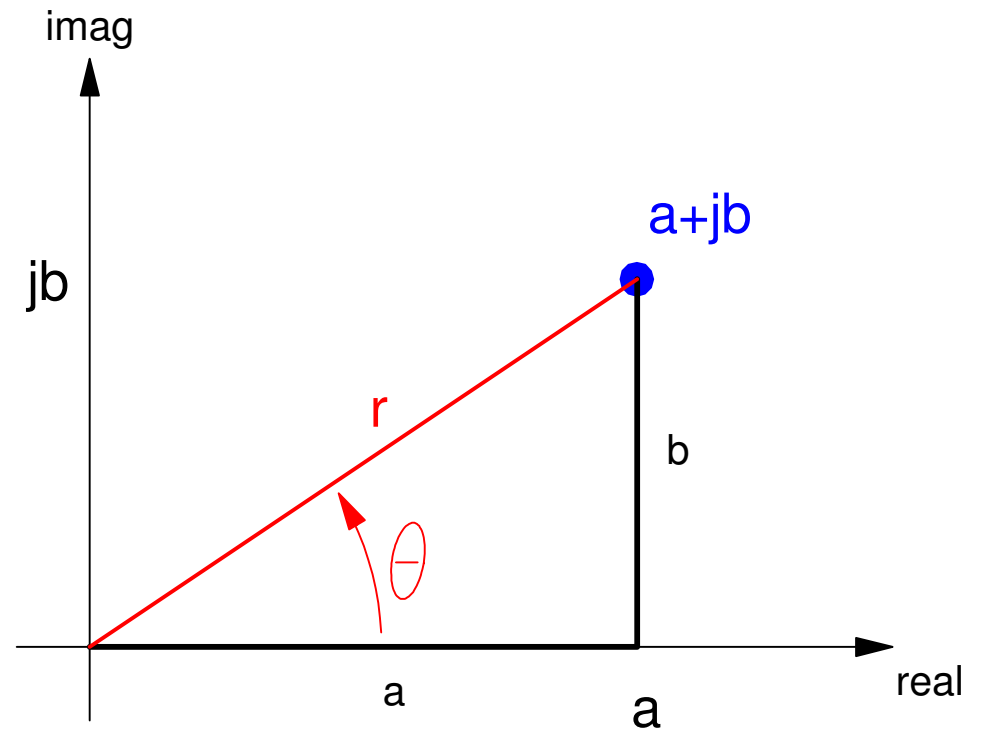
  $$X = r\angle\theta = r; e^{j\theta}$$

Conversion polar to rectangular:

$$a = r \cdot \cos \theta$$

$$b = r \cdot \sin \theta$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

A few more definitions:

$$real(a + jb) = a$$

$$imag(a + jb) = b$$

$$|r\angle\theta| = r = \sqrt{a^2 + b^2} \qquad magnitude$$

$$\angle(r\angle\theta) = \theta = \arctan\left(\frac{b}{a}\right) \qquad angle$$

$$(a + jb)^* = (a - jb) \qquad complex\ conjugate$$
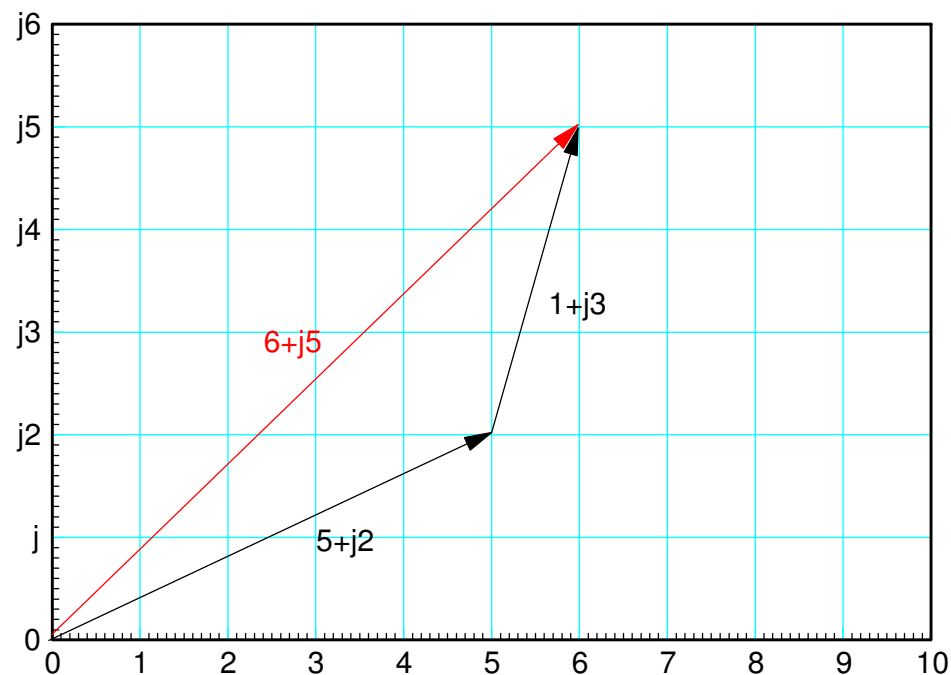
Note

$$(a + jb)(a - jb) = a^2 + b^2 = r^2$$

Algebra with Complex Numbers

- Real parts add, Complex parts add

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

$$(5 + j2) + (1 + j3) = 6 + j5$$

Multiplication:

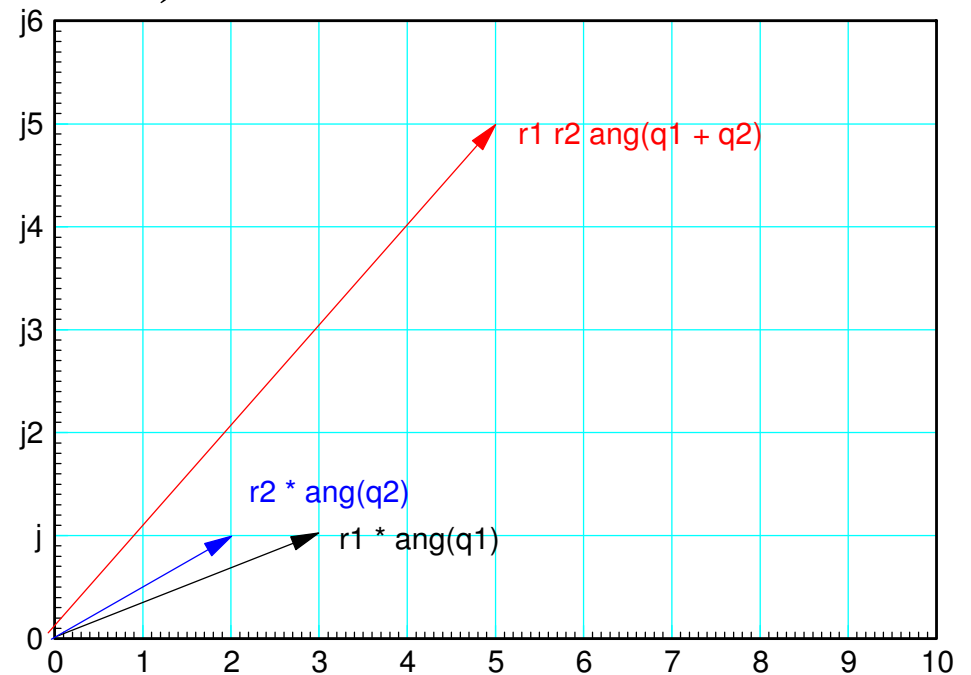$$(a + jb) \cdot (c + jd) = (ac - bd) + j(ad + bc)$$

$$(r_1 \angle \theta_1) \cdot (r_2 \angle \theta_2) = r_1 r_2 \angle (\theta_1 + \theta_2)$$

Example

$$(3 + j1)(2 + j1) = (3.16 \angle 18.4^0)(2.23 \angle 26.6^0)$$

$$= 7.07 \angle 45^0$$

$$= 5 + j5$$

Division:

Rectanglar Form

$$\left(\frac{a+jb}{c+jd}\right) = \left(\frac{a+jb}{c+jd}\right)\left(\frac{c-jd}{c-jd}\right)$$

$$= \left(\frac{(ac-bd)+j(bc-ad)}{c^2+d^2}\right)$$

$$= \left(\frac{ac-bd}{c^2+d^2}\right) + j\left(\frac{bc-ad}{c^2+d^2}\right)$$

Polar Form

$$\left(\frac{r_1\angle\theta_1}{r_2\angle\theta_2}\right) = \left(\frac{r_1}{r_2}\right)\angle(\theta_1-\theta_2)$$

Euler's Identity: From

$$e^{jx} = \cos(x) + j\sin(x)$$

you can derive Euler's Identity:

$$\cos(x) = \left(\frac{e^{jx}+e^{-jx}}{2}\right)$$

$$\sin(x) = \left(\frac{e^{jx}-e^{-jx}}{2j}\right)$$

A cosine function is composed of a complex exponential and its compex conjugate

- Important in communications
- A sine wave has two components: +jw and -jw
- Upper side band and lower side band

Phasors:

- Represent a sine wave with a single complex number, and
- Represent the impedance of capacitors and inductors for sinusoidal inputs.
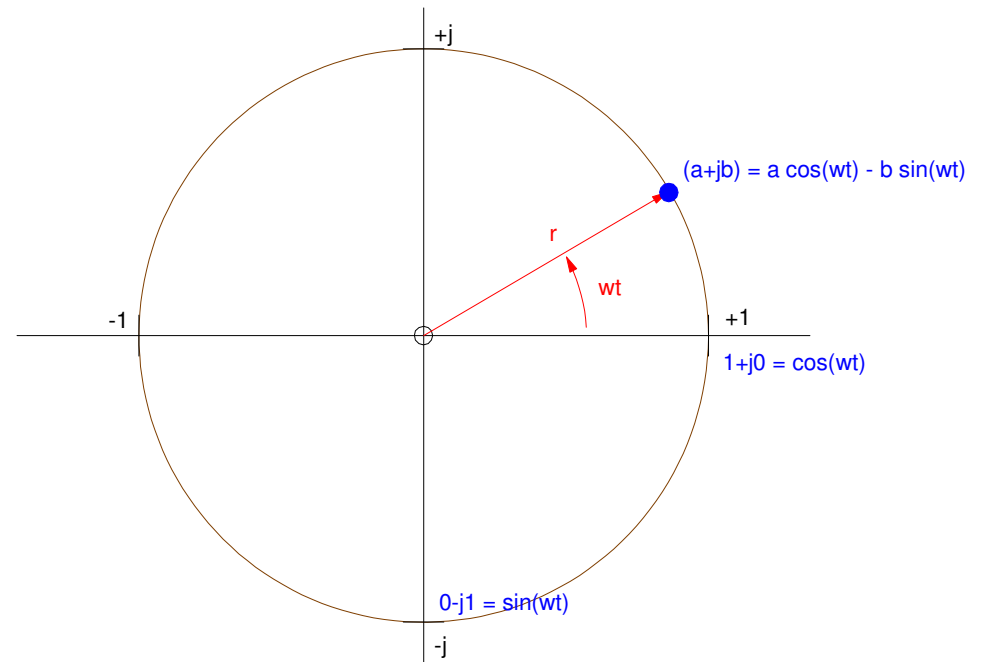
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$real(e^{j\omega t}) = \cos(\omega t)$$

$$1 \Longleftrightarrow \cos(\omega t)$$

$$(a + jb)e^{j\omega t} = (a + jb)(\cos(\omega t) + j\sin(\omega t))$$

$$= (a\cos(\omega t) - b\sin(\omega t)) + j(\cdots)$$

$$a + jb \Longleftrightarrow a\cos(\omega t) - b\sin(\omega t)$$

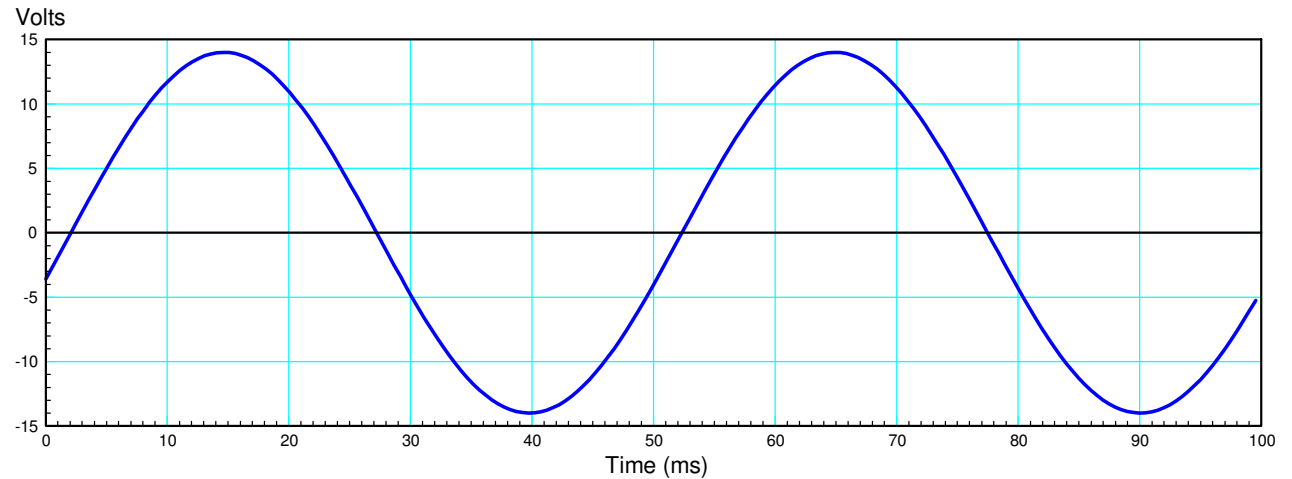$$r\angle\theta \Longleftrightarrow r\cos(\omega t + \theta)$$

Phasor Voltages

Rectangular Form

- Real = cosine
- -Imag = sine

Polar Form

- Amplitude = amplitude
- Delay = - phase

Example: Determine

- The frequency and
- The phasor representation for V
- (handout problem #1)

Complex Impedances

Resistors: $Z = R$

$$V = IR$$

Capacitors: $Z = \left(\dfrac{1}{j\omega C}\right)$

$$I = C\frac{dV}{dt} = C\frac{d}{dt}(e^{j\omega t})$$

$$I = j\omega C\ V$$
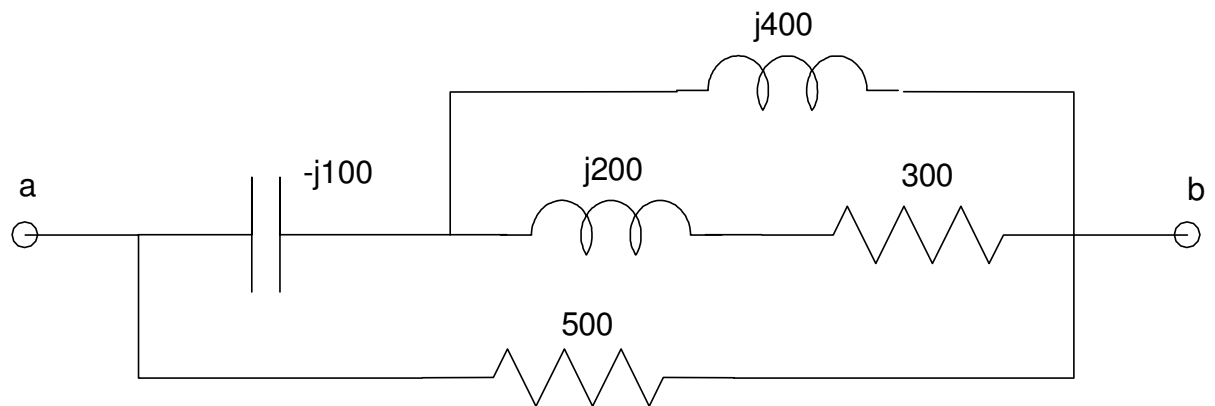
Inductors: $Z = j\omega L$

$$V = L\frac{dI}{dt} = L\frac{d}{dt}(e^{j\omega t})$$

Circuit Analysis with Phasors

- AC is just like DC analysis - only now with complex numbers

Example: Determine Rab

- handout problem #2

Example: Determine y(t)

$\omega = 20$ rad/sec

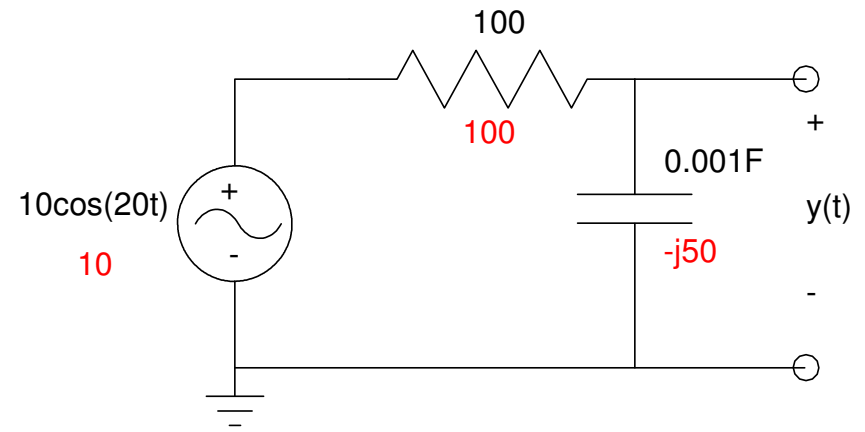$V_{in} \Rightarrow 10 + j0$

$C \Rightarrow \frac{1}{j\omega C} = -j50$

By voltage division

$Y = \left(\frac{-j50}{-j50+100} \right)(10 + j0)$

$Y = 2 - j4 = 4.472\angle -63.43^0$

$y(t) = 2\cos(20t) + 4\sin(20t)$

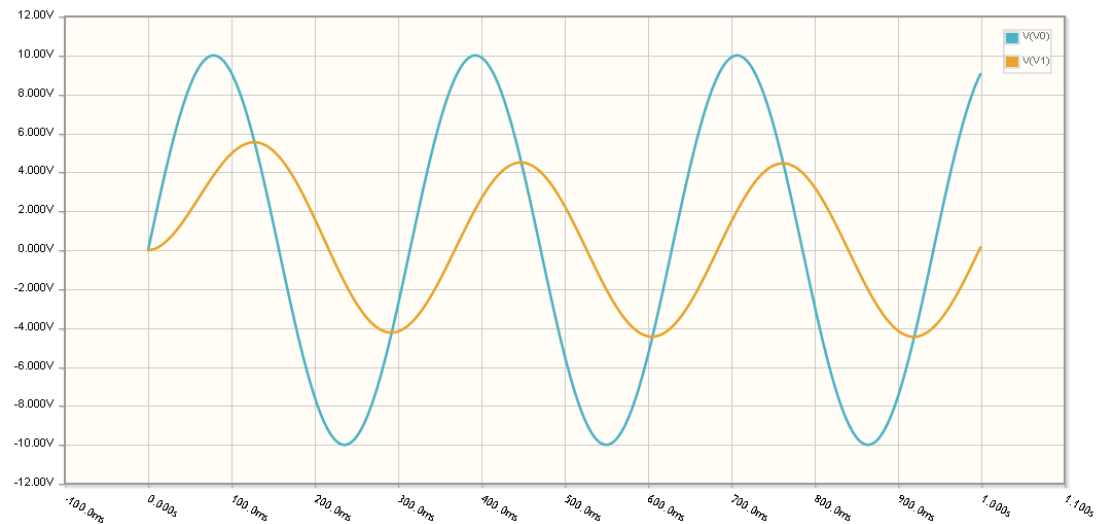$y(t) = 4.472\cos(20t - 63.43^0)$
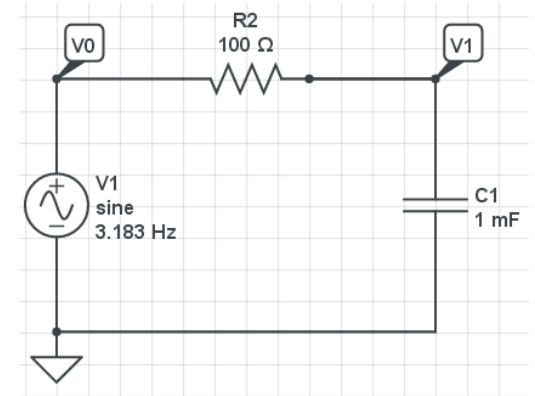
Checking in Circuitlab:

Output (Orange)

- Peak = 4.472V
- Delay = 17% of one cycles (63 degrees)

Note:

- Rectangular form is easier for calculations
- Polar form is easier in lab
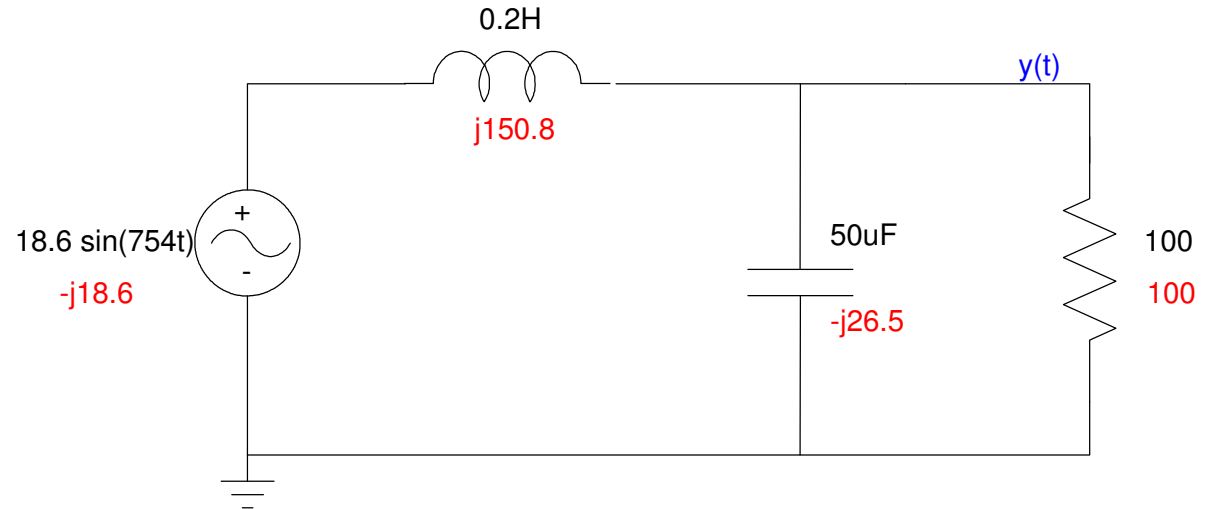
Both are equivalent

Example 2: Determine y(t)

$\omega = 754$ rad/sec

$V_{in} = 0 - j18.6$

$0.2H \Rightarrow j\omega L = j150.8\Omega$

$50\mu F \Rightarrow \dfrac{1}{j\omega C} = -26.5\Omega$

$-j26.5 \| 100 = \left(\dfrac{1}{-j26.5} + \dfrac{1}{100} \right)^{-1} = 25.64\angle -75.14^0$

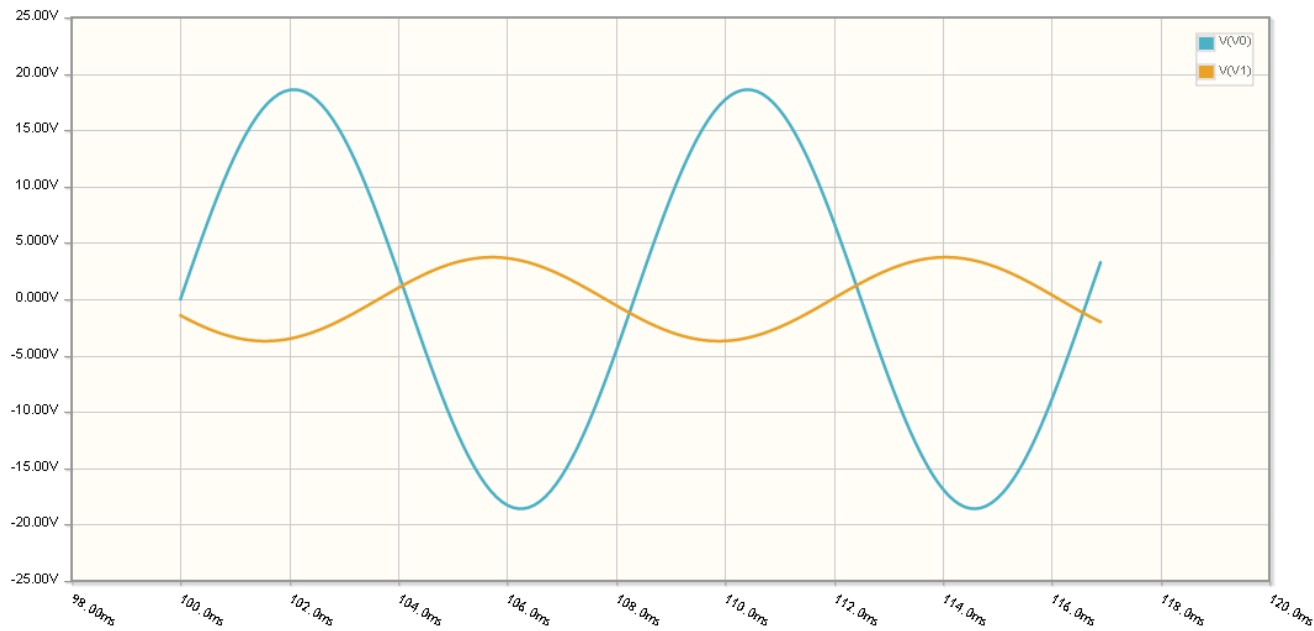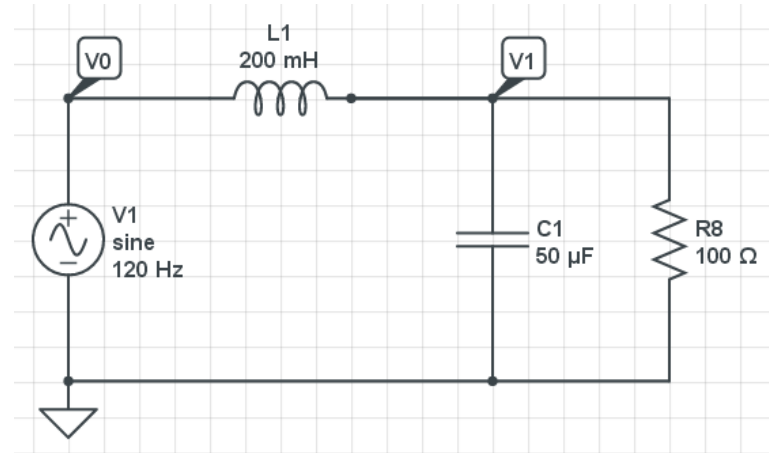$Y = \left(\dfrac{25.64\angle -75.14^0}{25.64\angle -75.14^0 + j150.8} \right) \cdot (-j18.6)$

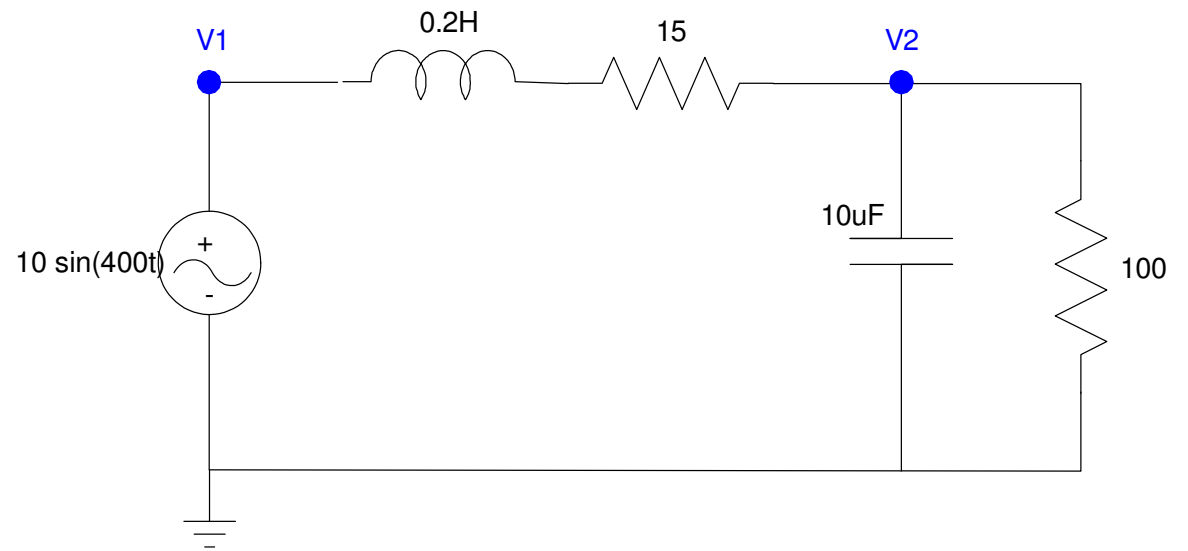$Y = 3.779\angle 107.8^0$

$y(t) = 3.779\cos{(754t + 107.8^0)}$

Checking in Circuitlab

- Orange = V1(t)
- Peak = 3.779V
- Delay = 70% of a perion (-252 degrees)

Example: Find V2(t)

- handout problem #3

Summary

AC analysis of circutis is just like DC analysis, only with complex numbers

Phasor Voltages

- real = cosine
- -imag = sine

Phasor impedances

- $R \rightarrow R$
- $L \rightarrow j\omega L$
- $C \rightarrow \frac{1}{j\omega C} = \frac{-j}{\omega C}$