Complex Numbers and Phasors

Complex Numbers:

Define

$$j = \sqrt{-1}$$

$$j^2 = -1$$

Also define the complex exponential:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

A complex number has two terms: a real part and a complex part:

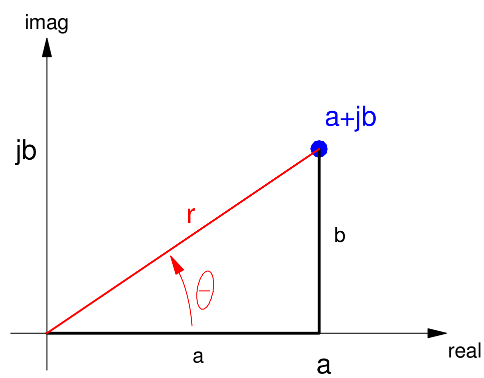
$$X = a + jb$$

You can also represent this in polar form:

$$X = r\angle\theta$$

which is short-hand notation for

$$X = r \cdot e^{j\theta}$$



Rectangular form (a + jb) and polar form $r\angle\theta$ of a complex number

You can convert from polar to rectangular form with

$$a = r \cdot \cos\theta$$

$$b = r \cdot \sin\theta$$

and from rectangular to polar form with

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

A few more definitions:

$$real(a+jb) = a$$

$$imag(a+jb) = b$$

$$|r\angle\theta| = r \qquad \text{(magnitude)}$$

$$\angle(r\angle\theta) = \theta \qquad \text{(angle)}$$

Complex Conjugate: Change the sign of the complex part

$$(a+jb)^* = (a-jb)$$

A number times its complex conjugate is the magnitude squared:

$$(a+jb)(a-jb) = a^2 - jab + jba - j^2b^2$$

$$= a^2 + b^2$$

$$= r^2$$

Complex Algebra:

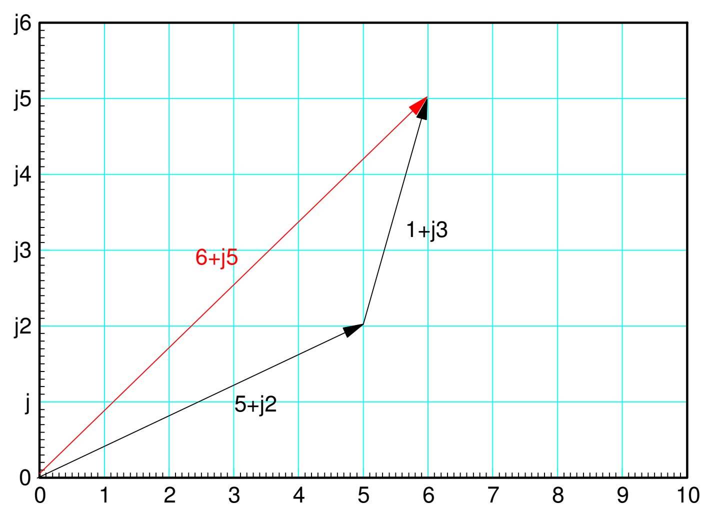
Addition and Subtraction: When adding complex numbers, the real part adds and the complex part adds:

$$(a+jb)+(c+jd) = (a+c)+j(b+d)$$

$$(a+jb)-(c+jd) = (a-c)+j(b-d)$$

Addition and subtraction is much easier in rectangular form. For example:

$$(5+j2)+(1+j3) = 6+j5$$



Addition of complex numbers

Multiplication: In rectangular form:

$$(a+jb)\cdot(c+jd) = ac+jad+jbc+j^2bd$$
$$= (ac-bd)+j(ad+bc)$$

In polar form:

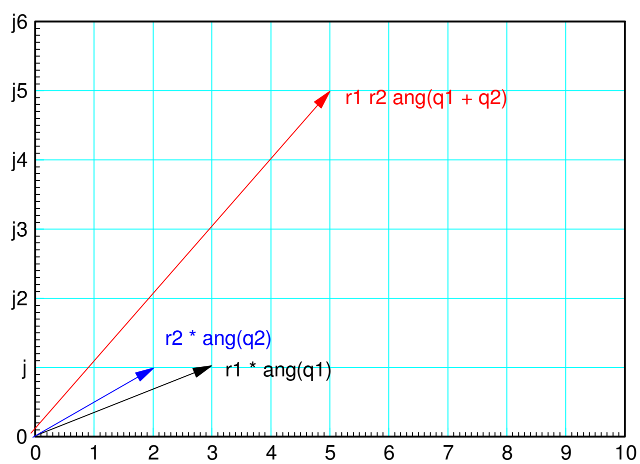
$$(r_1\angle\theta_1)\cdot(r_2\angle\theta_2) = r_1 r_2\angle(\theta_1+\theta_2)$$

Polar form is much simpler: when multiplying complex numebers:

- The magnitude multiplies
- The angles add

Example:

$$(3+j1)(2+j1) = (3.16\angle 18.4^0)(2.23\angle 26.6^0)$$
$$= 7.07\angle 45^0$$
$$= 5+j5$$



Multiplication

Division: In rectangular form:

$$\left(\frac{a+jb}{c+jd}\right) = \left(\frac{a+jb}{c+jd}\right)\left(\frac{c-jd}{c-jd}\right)$$
$$= \left(\frac{(ac-bd)+j(bc-ad)}{c^2+d^2}\right)$$
$$= \left(\frac{ac-bd}{c^2+d^2}\right)+j\left(\frac{bc-ad}{c^2+d^2}\right)$$

In polar form:

$$\left(\frac{r_1\angle\theta_1}{r_2\angle\theta_2}\right) = \left(\frac{r_1}{r_2}\right)\angle(\theta_1-\theta_2)$$

Polar form is much simpler: when dividing complex numebers:

- The magnitude divides

- The angles subtract

Example: Simplify the following

$$Y = \left(\frac{2s+3}{(s+3)(s+4)}\right)_{s=j3}$$

Solution: A calculator that does complex numbers really helps:

$$= \left(\frac{2(j3)+3}{(j3+3)(j3+4)}\right)$$

$$= \left(\frac{3+j6}{3+j21}\right)$$

$$= 0.3 - j0.1$$

Euler's Identity: From

$$e^{jx} = \cos(x) + j\sin(x)$$

you can derive Euler's Identity:

$$\cos(x) = \left(\frac{e^{jx}+e^{-jx}}{2}\right)$$

$$\sin(x) = \left(\frac{e^{jx}-e^{-jx}}{2j}\right)$$

A cosine function is composed of a complex exponential and its compex conjugate

This is important when you get to communications: when you broadcast an audio signal, you actually send two signals: the audio signal and its complex conjugate (requiring twice the bandwidth and twice the power).

Phasors:

Phasors are a way to

- Represent a sine wave with a single complex number, and
- Represent the impedance of capacitors and inductors for sinusoidal inputs.

Phasors and Sine Waves

From the definition of a complex exponential

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

you can represent a generalized sine wave as

$$(a + jb)e^{j\omega t} = (a + jb)(\cos(\omega t) + j\sin(\omega t))$$

$$= (a\cos(\omega t) - b\sin(\omega t)) + j(b\cos(\omega t) + a\sin(\omega t))$$

Taking the real part (the part we soon on an oscilloscope):

$$real((a + jb)e^{j\omega t}) = a\cos(\omega t) - b\sin(\omega t)$$

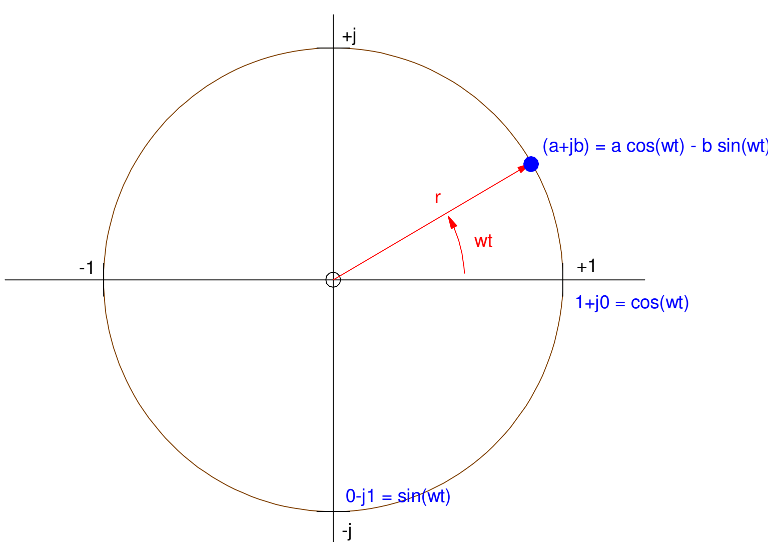
or the phasor representation of a sine wave is:

$$\boxed{a + jb \Rightarrow a\cos(\omega t) - b\sin(\omega t)}$$

Phasors represent a sinusoidal signal:

The real part is cos()

The imaginary part is -sin()



Phasor representation of a sine wave: e^{jwt} is a complex number that spins around the origin. The real part of e^{jwt} is a cosine wave

If you prefer polar form:

$$r\angle\theta \cdot e^{j\omega t} = re^{j\theta} \cdot e^{j\omega t}$$

$$= r \cdot e^{j(\omega t+\theta)}$$

$$= r\cos(\omega t + \theta) + jr\sin(\omega t + \theta)$$

Taking the real part:

$$r\angle\theta \Rightarrow r\cos(\omega t + \theta)$$

Phasors represent a sinusoidal signal:

The magnitude is the magnitude of the cosine wave

The angle is the phase shift of the cosine wave

Note that when using phasors, frequency is understood: you have to specify what frequency you're dealing with as a footnote.

Complex Impedances

At DC circuits, you have

V = I R

For AC circuits, it gets a little more complicated: you have

$$V = IR$$

$$V = L\frac{dI}{dt}$$

$$I = C\frac{dV}{dt}$$

If we could convert capacitors and inductors into something that looks like a resistor

$$V = IZ$$

then we could use all of the techniques taught in Circuits I. Phasors are a way to do that.

First, assume all signals (voltages and currents) are in the form of e^{jwt} (i.e. they are sinusoids). Then for resistors:

$$V = IR = IZ$$

The complex impedance of a resistor is R

For inductors:

$$V = L\frac{dI}{dt}$$

Assuming

$$I = e^{j\omega t}$$

$$V = L \cdot j\omega \cdot e^{j\omega t}$$

$$= (j\omega L) \cdot I$$

$$= Z \cdot I$$

The complex impedance of an inductor is jwL

For a capacitor

$$I = C\frac{dV}{dt}$$

Assuming

$$V = e^{j\omega t}$$

$$I = C \cdot j\omega \cdot e^{j\omega t}$$

$$= j\omega C \cdot V$$

$$V = \left(\frac{1}{j\omega C}\right) \cdot I$$

$$= Z \cdot I$$

The complex impedance of a capacitor is 1 / jwC

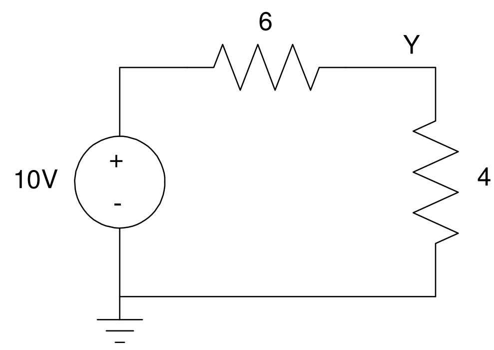
In summary:

Parameter		Phasor Representation
Voltage	V = a cos(wt) + b sin(wt)	a - jb
Resistor	R	R
Inductor	L	jwL
Capacitor	C	1 / jwC

Circuit Analysis with Phasors

With phasors, you can analyze RLC circuits with sinusoidal inputs just like you did with DC circuits - ableit using complex numbers.

Example 1: Determine y(t)



Example 1: Solve for y(t) with a DC input

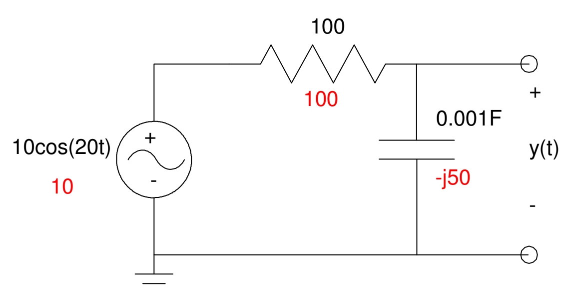
This is a trick question: At DC you don't need to use phasors - all signals will be real. Using voltage division

$$Y = \left(\frac{4}{4+6}\right) \cdot 10$$

$$Y = 4$$

$$y(t) = 4$$

Example 2: Determine y(t)



Example 2: Find y(t) using phasor analysis.

First, convert to phasors and complex impedances (shown in red).

$$\omega = 20 \text{ rad/sec}$$

$$10\cos(20t) \Rightarrow (10 + j0)V$$

$$100\Omega \Rightarrow 100\Omega$$

$$0.001F \Rightarrow \frac{1}{j\omega C} = -j50\Omega$$

Now solve using votlage division:

$$Y = \left(\frac{-j50}{-j50+100}\right)10$$

$$Y = 2 - j4 \qquad \textit{rectangular form}$$

$$Y = 4.472\angle -63.43^0 \qquad \textit{polar form}$$

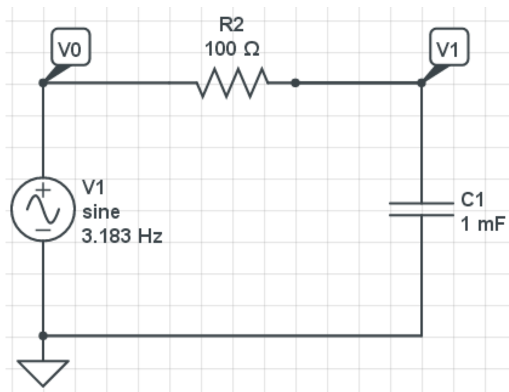
meaning (recall that ω = 20)

$$y(t) = 2\cos(20t) + 4\sin(20t) \qquad \textit{rectangular form}$$

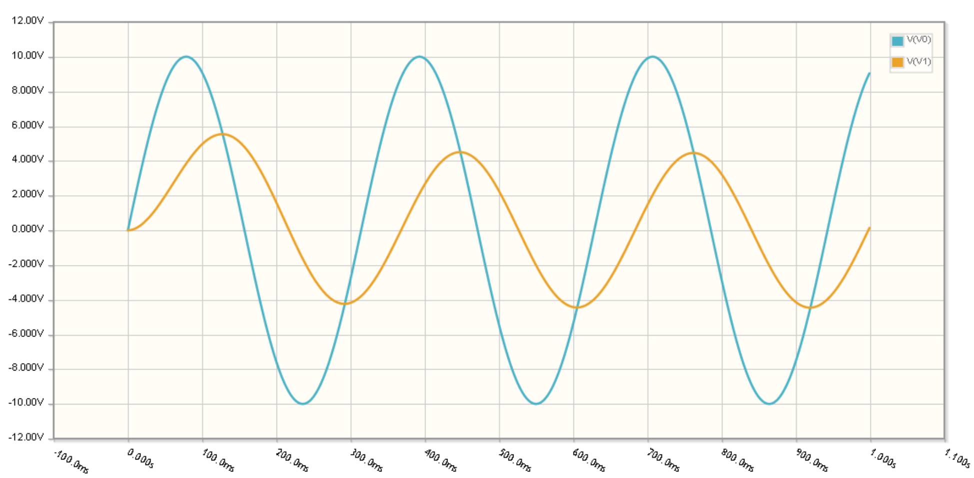
$$y(t) = 4.472\cos(20t - 63.43^0) \qquad \textit{polar form}$$

Both answers are correct - it's a matter of taste which form you prefer.

Checking in Circuitlab:

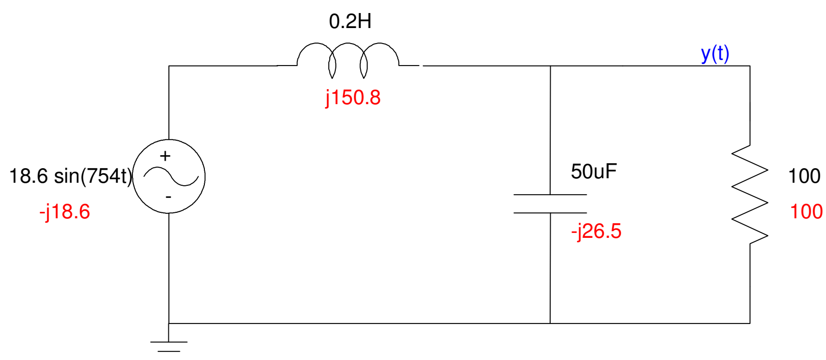


Circuitlab Circuit: Note that 20 rad/sec = 3.183 Hz



Circuitlab Transient Simulation: y(t) is shown in orange
Note that the the peak voltage of y(t) is 4.518V (vs. 4.472 computed), delayed by 17% of one cycle (63 degrees)

Example 2: Determine y(t)



Determine y(t) for a 18.6V, 120Hz input.

First, convert to phasors and complex impedances (shown in red)

$$\omega = 754 \text{ rad/sec}$$

$$0.2H \Rightarrow j\omega L = j150.8\Omega$$

$$50\mu F \Rightarrow \frac{1}{j\omega C} = -26.5\Omega$$

Solve using circuits techniques. The resistor and capacitor add in parallel:

$$-j26.5 || 100 = \left(\frac{1}{-j26.5} + \frac{1}{100}\right)^{-1} = 25.64\angle -75.14^0$$

Using voltage division

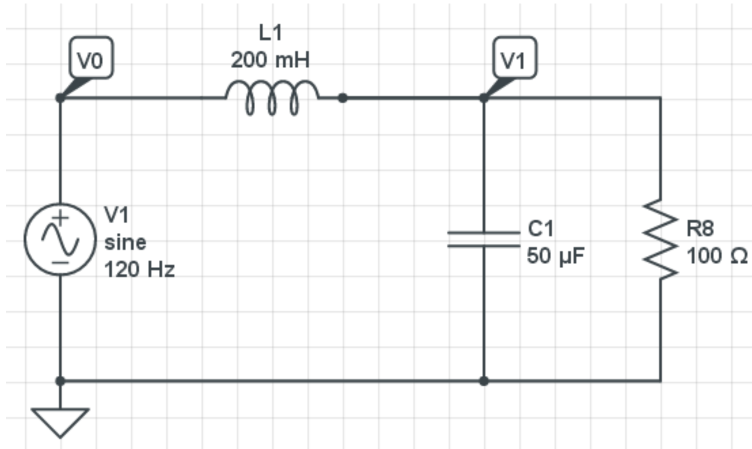
$$Y = \left(\frac{25.64\angle -75.14^0}{25.64\angle -75.14^0 + j150.8}\right) \cdot (-j18.6)$$

$$Y = 3.779\angle 107.8^0$$

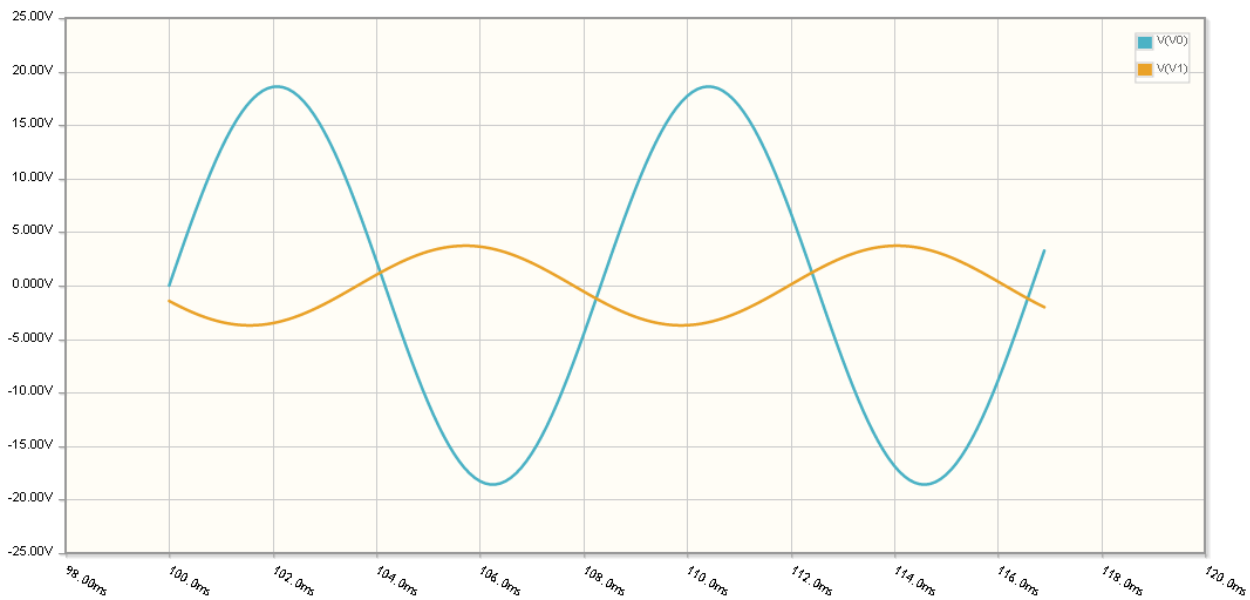
meaning (recall w = 754)

$$y(t) = 3.779\cos(754t + 107.8^0)$$

Checking in Circuitlab



Circuitlab Circuit: 18.6V peak, 120Hz sine wave input



Circuitlab Simulation. V1 (orange)
Vout = 3.659V peak (vs. 3.779V compted), delayed by +107 degrees