
Thevenin Equivalents & Max Power Transfer

ECE 211 Circuits I Lecture #27

Please visit [Bison Academy](#) for corresponding
lecture notes, homework sets, and solutions

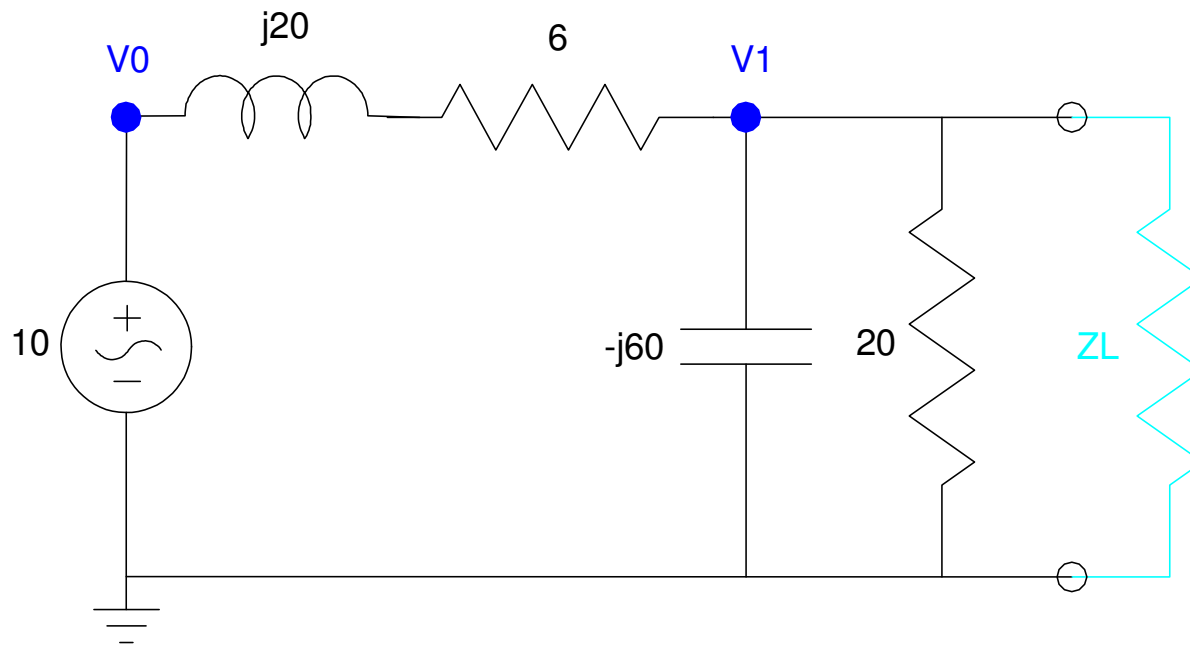
Thevenin Equivalents

Thevenin equivalents also work with phasors - only you get complex numbers for the Thevenin voltage and Thevenin resistance.

	VI relationship	Phasor Notation
Voltage	$v(t) = a \cos(\omega t) + b \sin(\omega t)$	$V = a - jb$
Resistor	$v = iR$	$Z_R = R$
Inductor	$v = L \frac{di}{dt}$	$Z_L = j\omega L$
Capacitor	$i = C \frac{dv}{dt}$	$Z_C = \frac{1}{j\omega C}$

Example 1: Determine

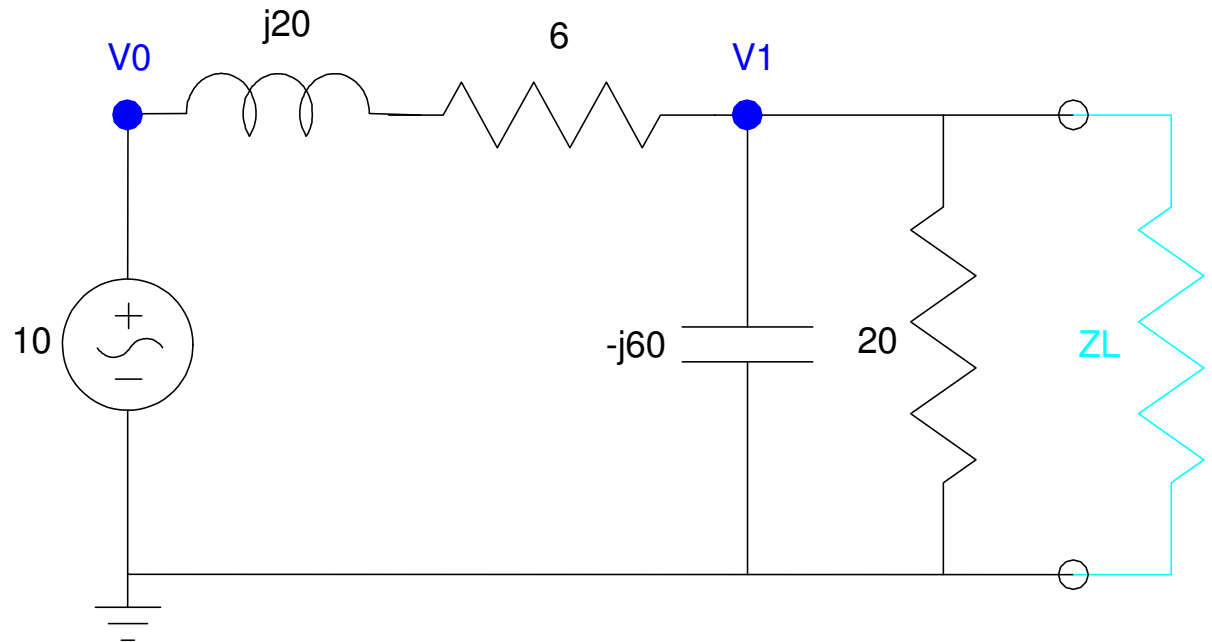
- The Thevenin equivalent for the following circuit,
- Z_L for max power transfer, and
- The maximum power to a load



Thevenin Equivalent

Find two of the following:

- Open-circuit voltage
- Short-circuit current
- Equivalent resistance



Open-Circuit Voltage

Voltage Nodes

$$\left(\frac{V_1 - V_0}{6 + j20}\right) + \left(\frac{V_1}{-j60}\right) + \left(\frac{V_1}{20}\right) = 0$$

Voltage division

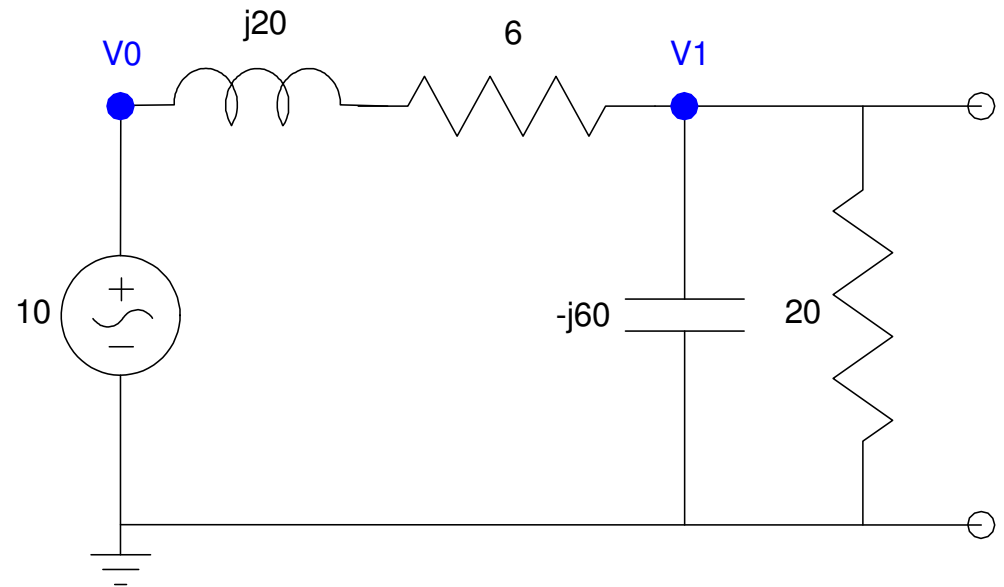
$$-j60 \parallel 20 = 18 - j6$$

$$V_1 = \left(\frac{(18 - j6)}{(18 - j6) + (6 + j20)}\right) 10$$

Result

$$V_1 = 4.508 - j5.129$$

This is V_{th}



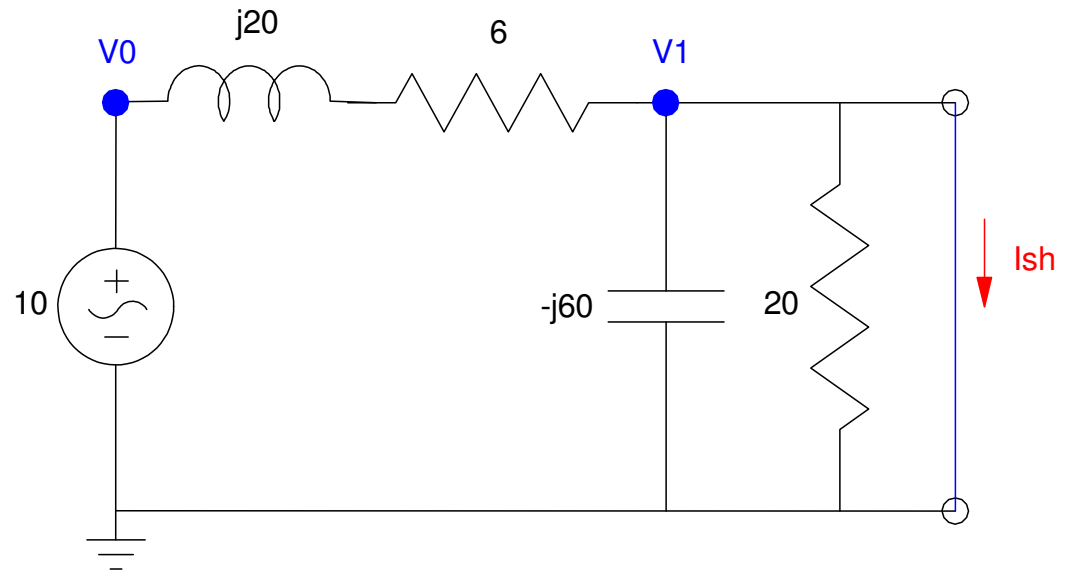
Short-Circuit Current

- Short the output
- Compute the current

$$I_{sh} = \left(\frac{10}{6+j20} \right)$$

$$I_{sh} = 0.138 - j0.459$$

This is I(Norton)



Thevenin Impedance

Option 1: $Z_{th} = V(\text{open}) / I(\text{short})$

$$Z_{th} = \left(\frac{4.5078 - j5.1295}{0.1376 - j0.4587} \right)$$

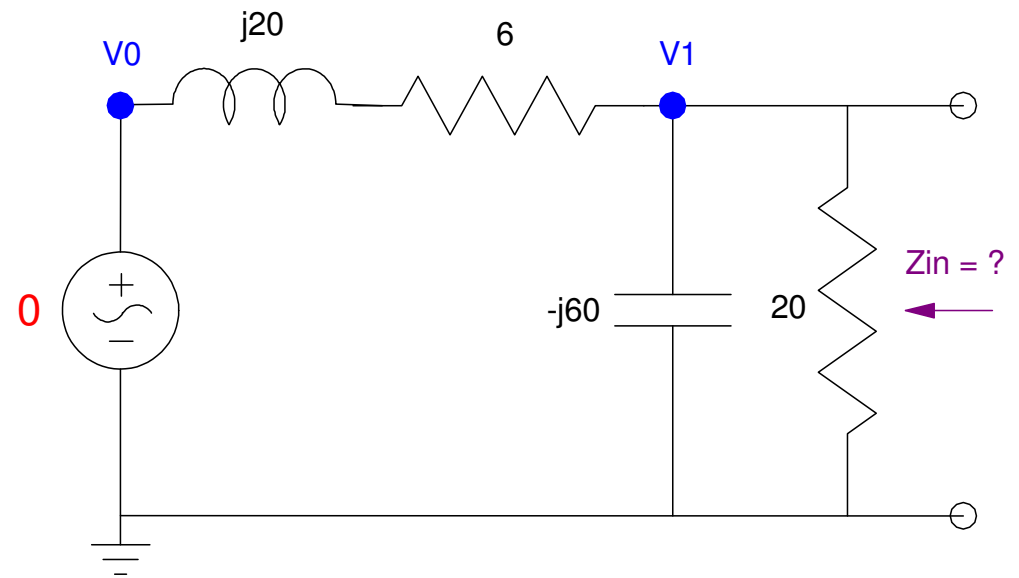
$$Z_{th} = 12.9637 + j5.9378$$

Option 2:

- Turn off the source
- Compute the impedance looking in

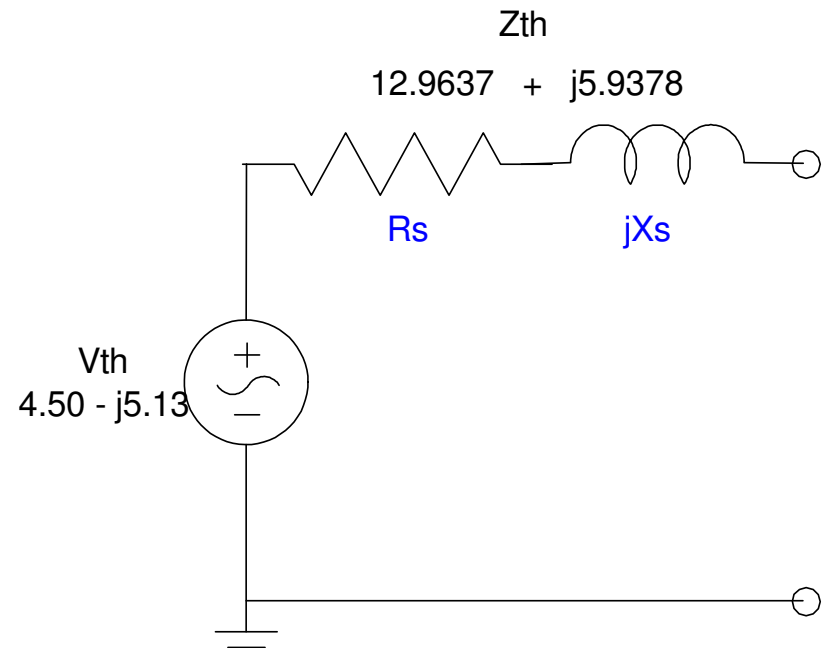
$$Z_{in} = (6 + j20) \parallel (-j60) \parallel (20)$$

$$Z_{in} = 12.9637 + j5.9378$$



Thevenin Equivalent

- V_{th} is the open-loop voltage
- Z_{th} is the Thevenin resistance
 - Series model
 - $R_s + jX_s$



Norton Equivalent

- I_n is the short-circuit current
- $Z_n = Z_{th}$
 - Parallel model
 - $R_p \parallel jX_p$

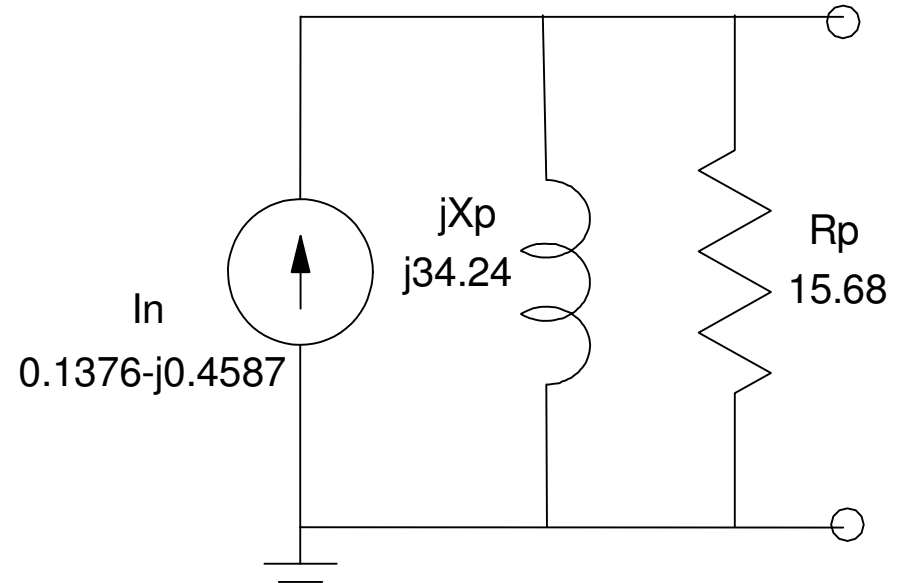
Note

$$R_s + jX_s = R_p \parallel jX_p$$

$$(R_s + jX_s) = \left(\frac{1}{R_p} + \frac{1}{jX_p} \right)^{-1}$$

$$\left(\frac{1}{R_s + jX_s} \right) = \frac{1}{R_p} + \frac{1}{jX_p}$$

$$12.9637 + j5.9378 = (15.6835) \parallel (j34.2408)$$

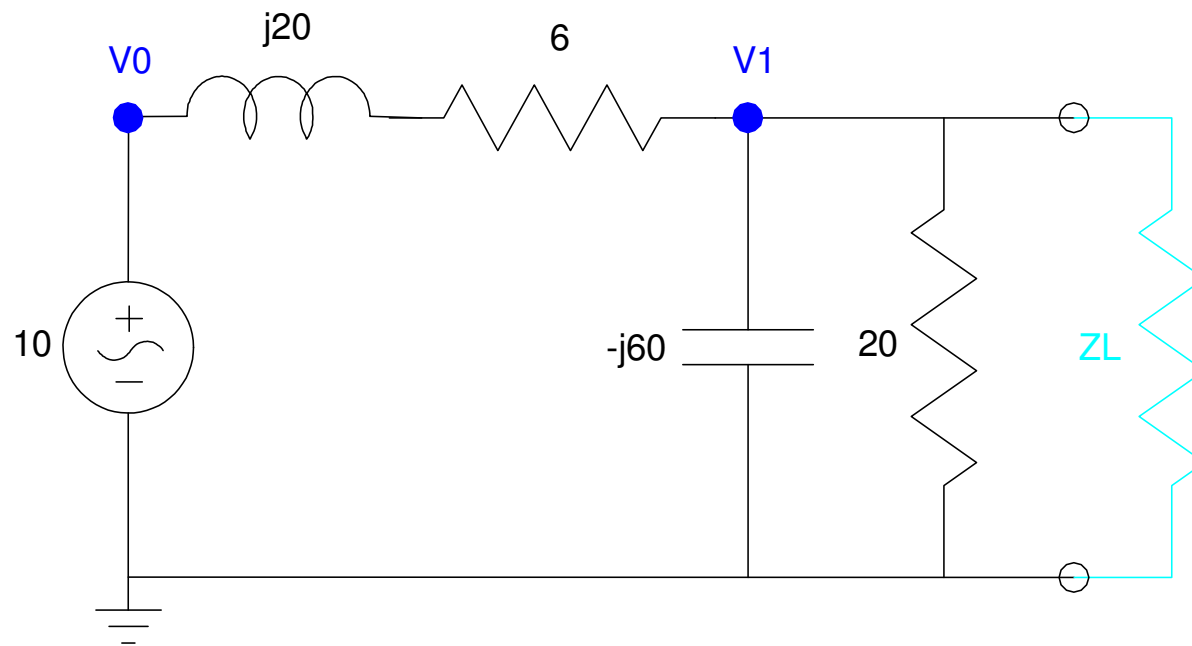


Max Power Transfer

- New Topic

What value of Z_L maximizes the power to the load?

What is the maximum power you can transfer?



Power Calculations

Equations for power are slightly different DC vs. AC

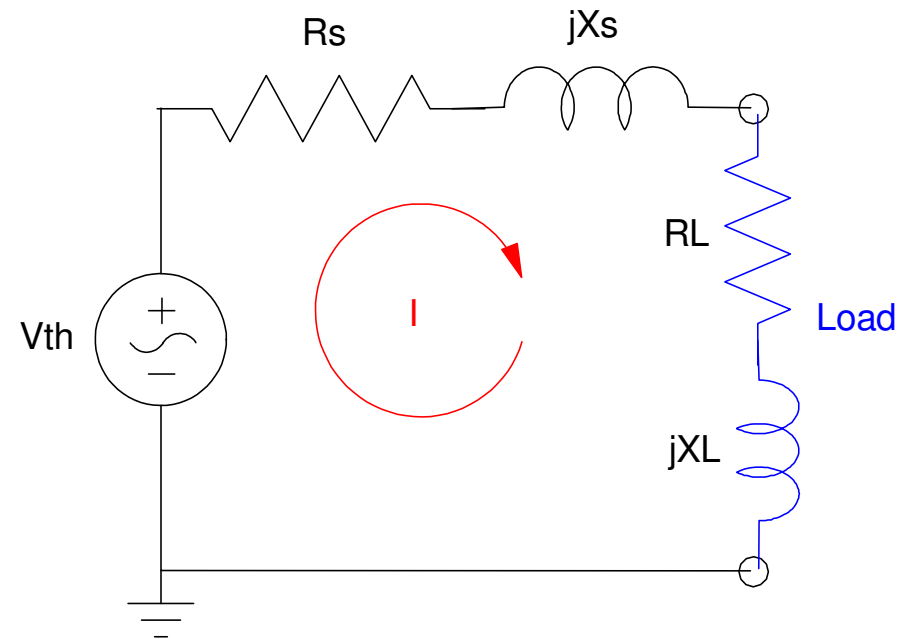
- The real part of P is the work done (or heat produced),
- The complex part of P is the energy that bounces back and forth.

DC	AC rms	AC peak
$P = V \cdot I$	$P = V_{rms} \cdot I_{rms}^*$	$P = \frac{1}{2} V_p \cdot I_p^*$
$P = I^2 R$	$P = I_{rms} ^2 \cdot Z$	$P = \frac{1}{2} \cdot I_p Z$
$P = \frac{V^2}{R}$	$P = \frac{ V_{rms} ^2}{Z^*}$	$P = \frac{1}{2} \cdot \frac{ V_{rms} ^2}{Z^*}$

Maximum Power to the Load

Procedure

- Find the Thevenin equivalent
- Find the power to the load
- Maximize the real part



Calculations

Voltage (rms units)

$$V_L = \left(\frac{R_L + jX_L}{(R_L + jX_L) + (R_s + jX_s)} \right) V_{th}$$

Current (rms units)

$$I_L = \left(\frac{V_{th}}{(R_L + jX_L) + (R_s + jX_s)} \right)$$

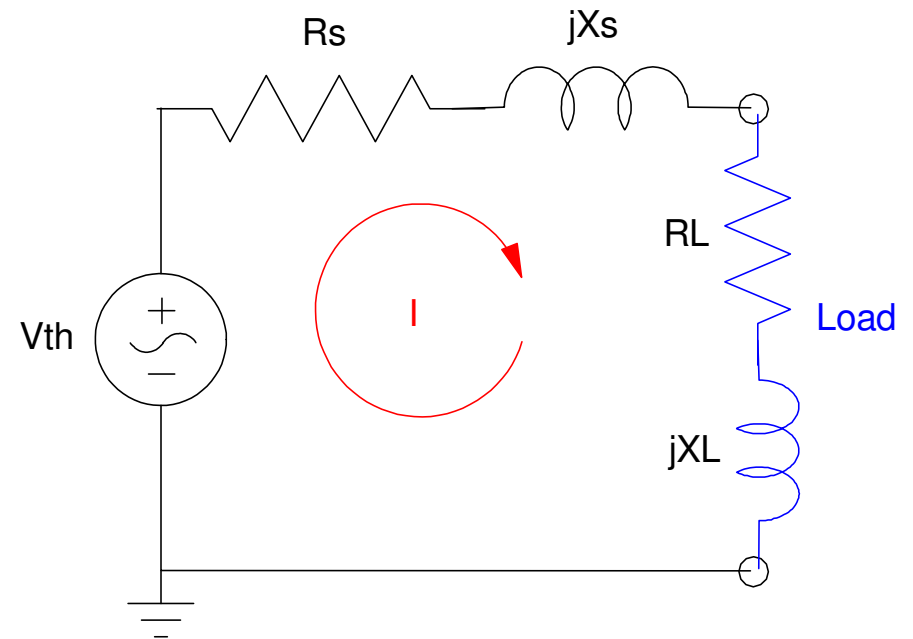
Power

$$P = V_L \cdot I_L^*$$

$$P = \left(\frac{R_L + jX_L}{(R_L + R_s)^2 + (X_L + X_s)^2} \right) |V_{th}|^2$$

Real part of power

$$P = \left(\frac{R_L}{(R_L + R_s)^2 + (X_L + X_s)^2} \right) |V_{th}|^2$$



Maximize power with respect to jX_L

$$P = \left(\frac{R_L}{(R_L + R_s)^2 + (X_L + X_s)^2} \right) |V_{th}|^2$$

$$\frac{dP}{dX_L} = 0 = \left(\frac{-R_L(2)(X_L + X_s)}{\left((R_L + R_s)^2 + (X_L + X_s)^2 \right)^2} \right)$$

$$X_L = -X_s$$

Make the complex impedance the negative of the complex part of the Thevenin series impedance

Maximize power with respect to R_L

$$P = \left(\frac{R_L}{(R_L + R_s)^2 + (X_L + X_s)^2} \right) |V_{th}|^2$$

$$P = \left(\frac{R_L}{(R_L + R_s)^2} \right) |V_{th}|^2$$

Maximize

$$\frac{dP}{dR_L} = 0 = \left(\frac{(R_L + R_s)^2 - R_L(2)(R_L + R_s)}{(R_L + R_s)^2} \right) |V_{th}|^2$$

$$(R_L + R_s)^2 - R_L(2)(R_L + R_s) = 0$$

$$(R_L + R_s)(R_s - R_L) = 0$$

Solution:

$$R_L = R_s$$

Net Result

$$Z_L = Z_{th}^*$$

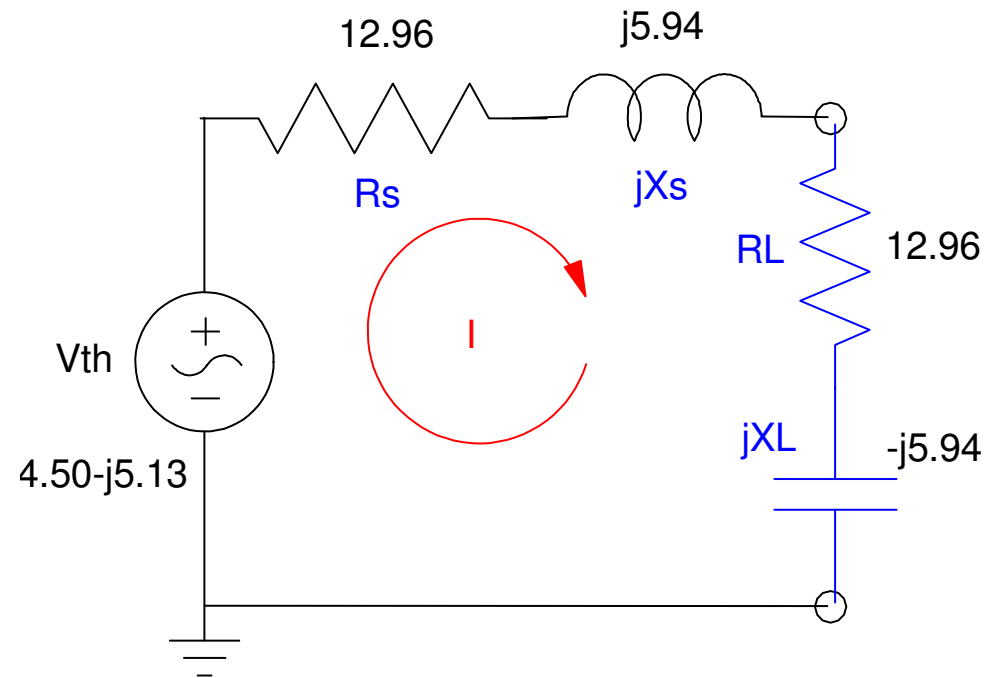
$$R_L + jX_L = R_s - jX_s$$

Max power is then

$$I = \frac{V_{th}}{R_s + R_L} = \frac{V_{th}}{2R_s}$$

$$P = V \cdot I^*$$

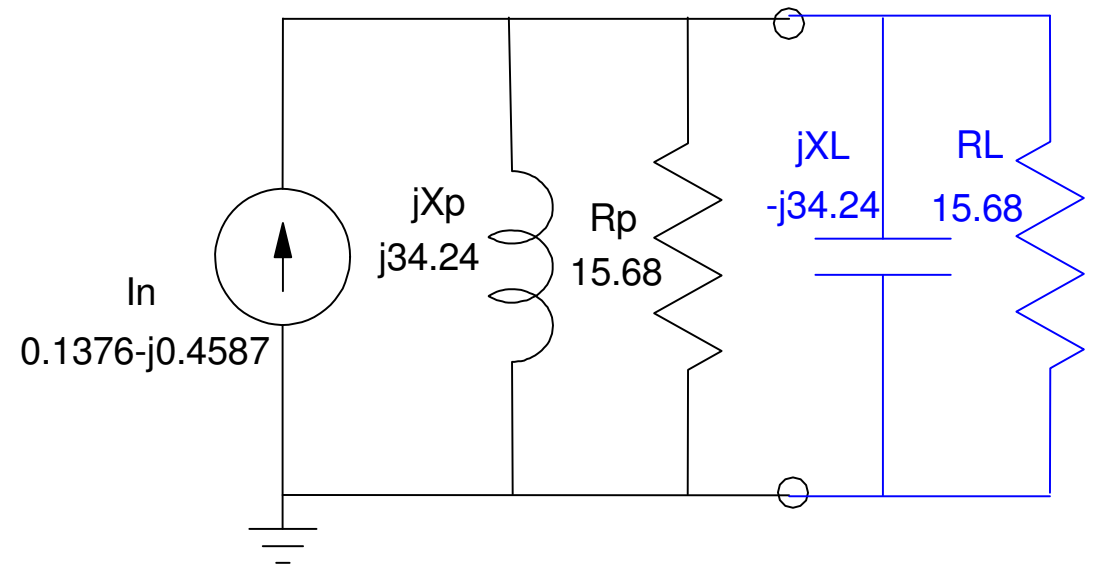
$$P = \frac{|V_{th}|^2}{2R_s}$$



Norton Model

You can also use the parallel model

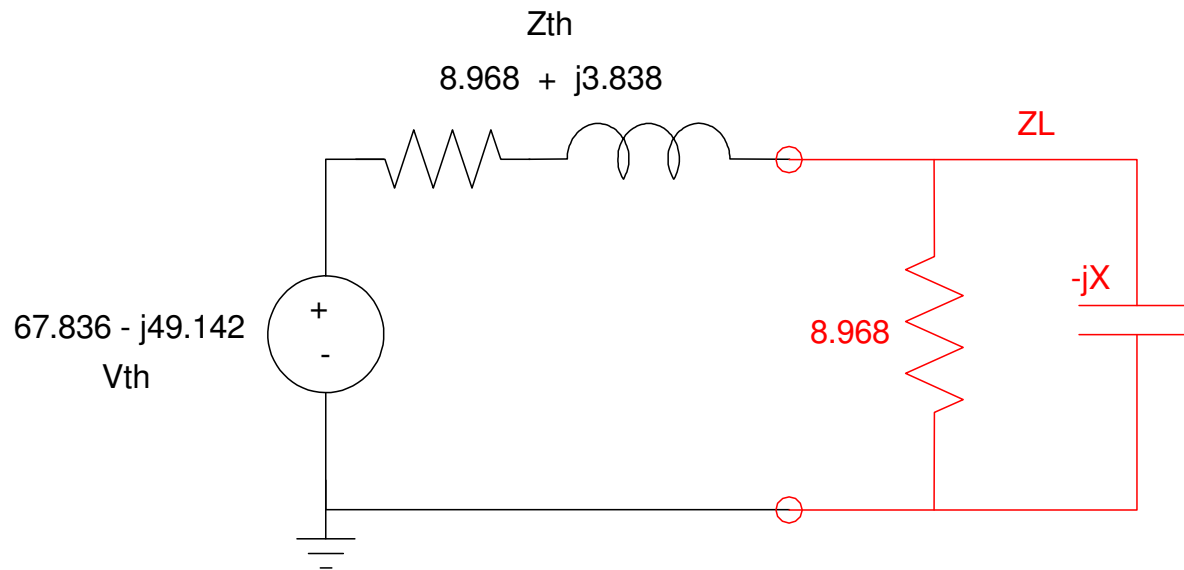
- $jX_L = -jX_p$
- $R_L = R_p$



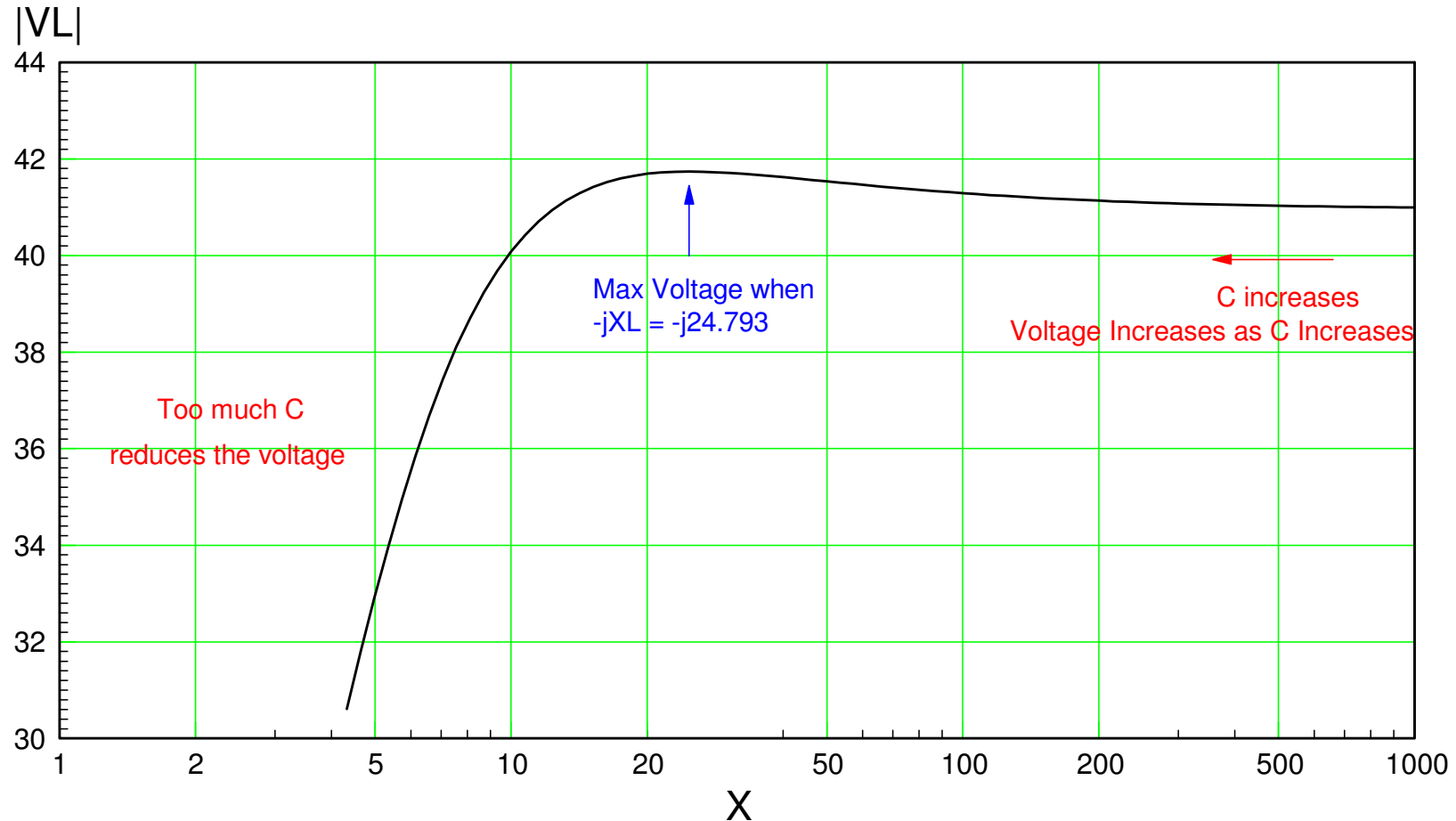
"Capacitors add Voltage"

If Z_{th} is inductive, then adding capacitors to the load,

- Cancels the complex part of Z_{th} , which
- Reduces the overall impedance, which
- Increases the current to the load, which
- Increases the voltage at the load.

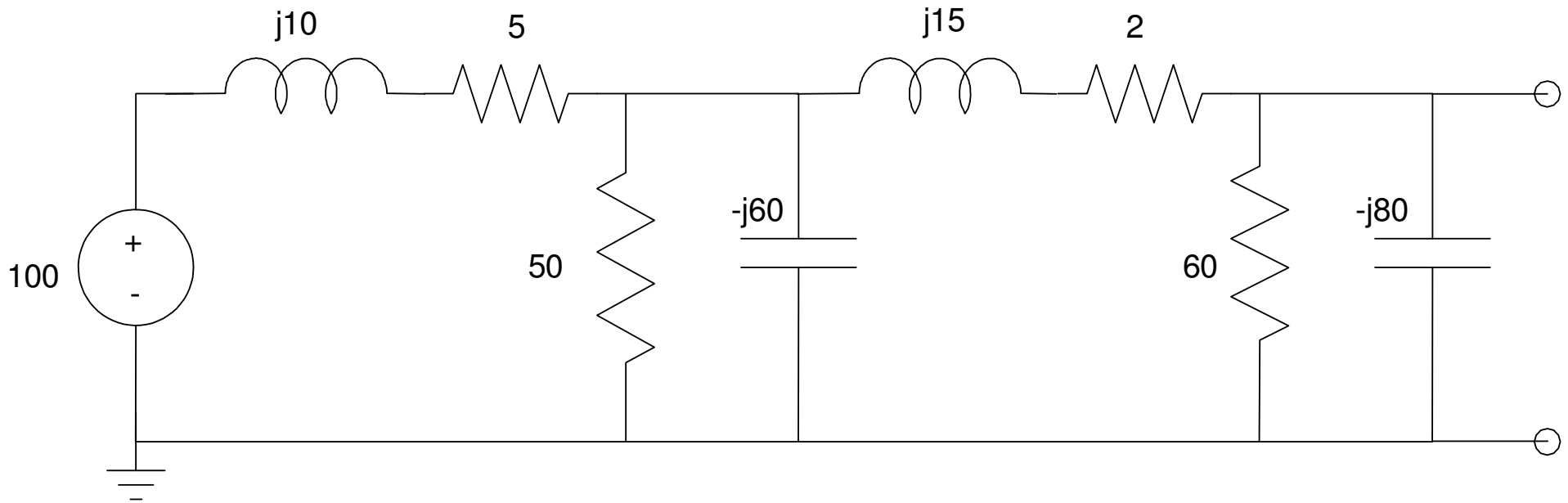


This only works up to a point: once you have cancelled all of the inductance ($+jX$), adding more capacitors will actually reduce the voltage.



Example 3: Source Transformations

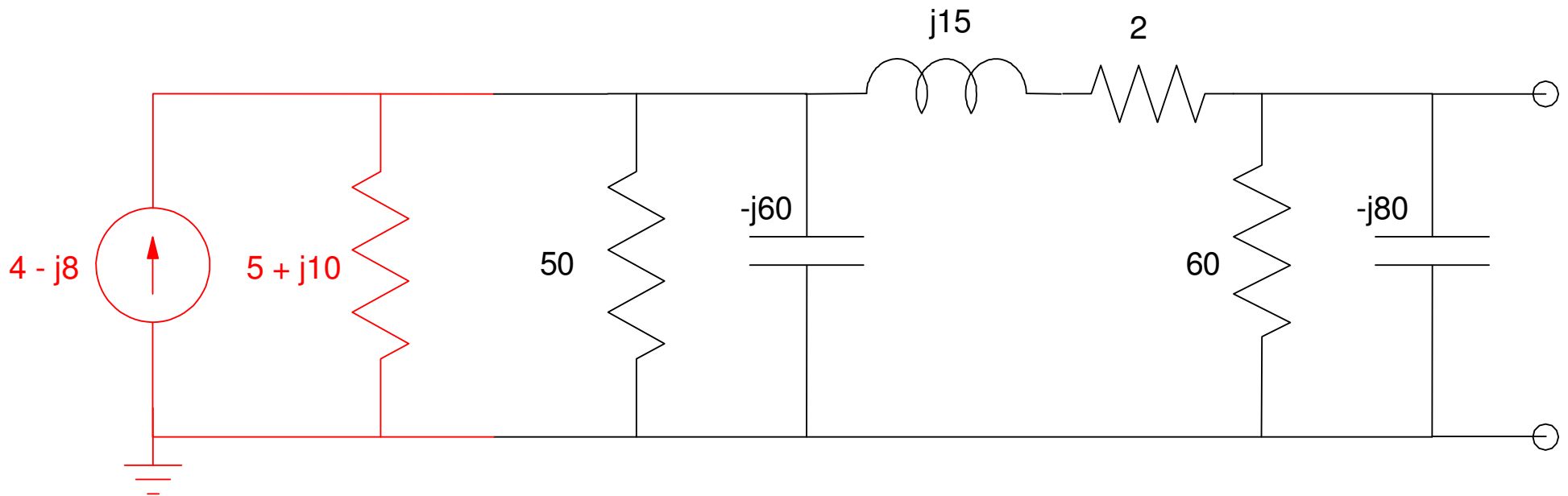
Source transformations also work with complex numbers. For example, determine the Thevenin equivalent for the following circuit:



Step 1: Convert to a Norton equivalent

$$Z_N = Z_{th} = 5 + j10$$

$$I_N = \frac{V_{th}}{Z_{th}} = 4 - j8$$



Combine impedances in parallel

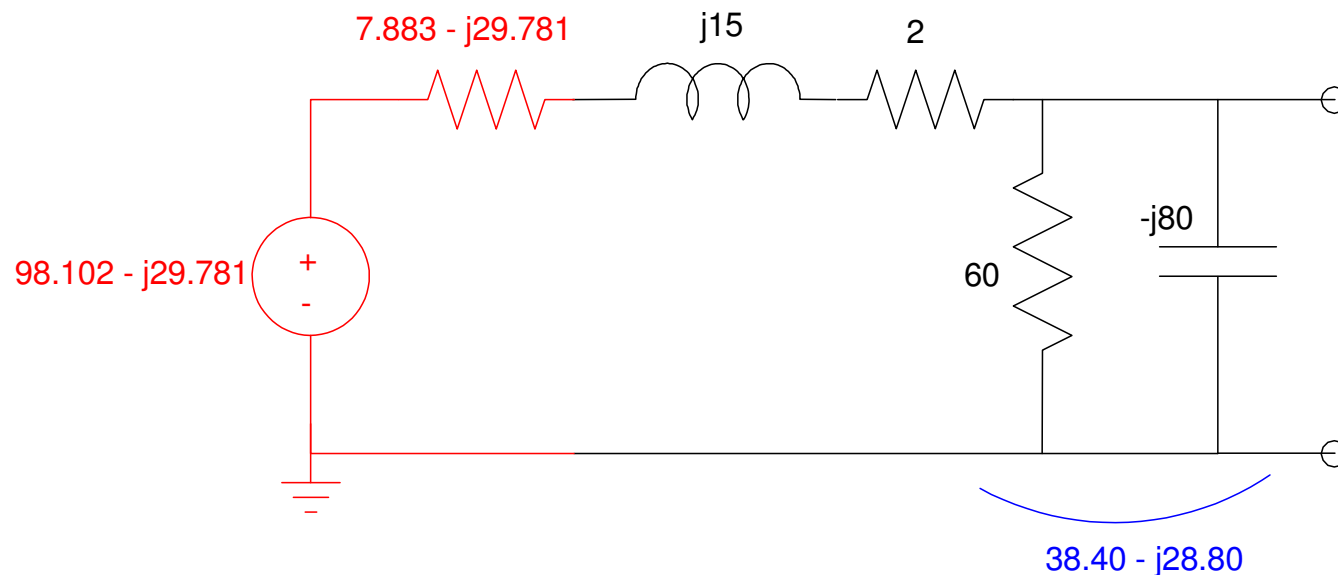
$$(5 + j10) \parallel (50) \parallel (-j60) = 7.883 + j8.321$$

Convert to Thevenin

$$Z_{th} = Z_N = 7.883 + j8.321$$

$$V_{th} = I_N \cdot Z_N = (4 - j8) \cdot (7.883 + j8.321)$$

$$V_{th} = 98.102 - j29.781$$



Now find the Thevenin equivalent.

By voltage division:

$$V_{th} = \left(\frac{(38.40 - j28.80)}{(38.40 - j28.80) + (7.883 - j29.781) + (2 + j15)} \right) (98.102 - j29.781)$$

$$V_{th} = 68.701 - j74.405$$

Zth: Turn off the source and measure the impedance

$$Z_{th} = ((7.83 - j29.781) + (2 + j15)) || (60) || (-j80)$$

$$Z_{th} = 20.077 + j14.931$$

Summary

RLC circuits can be reduced to their Thevenin and Norton equivalent

- Same procedure as DC
- Using phasors and complex numbers

Max power transfer is when

- Series Connection
 - $R_L = R_s$
 - $jX_L = -jX_s$
- Parallel Connection
 - $R_L = R_p$
 - $jX_L = -jX_p$