

Natural Response of an RL Circuit

Introduction

In the previous lecture, the natural response for 1-stage and 2-stage RC circuits was found. In this lecture, the same procedure will be used to find the natural response of 1-stage and 2-stage RL circuits such as presented in Figure 1.

The procedure will be similar to what was used with RC circuits:

- First, find the initial conditions at $t=0$.
- Next, find the differential equations that describe the circuit.
- Then solve.

The resulting differential equation will be solved using CircuitLab, calculus with the p-operator, and using eigenvalues and eigenvectors.

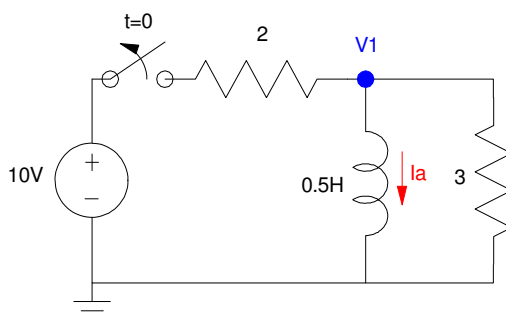


Figure 1: Find the natural response of a 1-stage RL circuit

Note that there is a slight difference between inductors and capacitors. For capacitors, the energy stored is related to the voltage across the capacitor

$$E = \frac{1}{2}CV^2$$

which results in the VI characteristic of capacitors being

$$I = C\frac{dV}{dt}$$

For inductors, the energy stored is related to the current through the inductor

$$E = \frac{1}{2}LI^2$$

This results in the VI characteristics for inductors being

$$V = L\frac{dI}{dt}$$

This means that

- Each inductor adds a 1st-order differential equation to the dynamic of the circuit, and
- The differential equation is in terms of current rather than voltage.

Or, at least, it's more straightforward to write the differential equations in terms of current for inductors.

1-Stage RL Circuit

Let's start with the 1-stage RL circuit of Figure 1. For this circuit, find

- The initial condition (the current I_a at $t=0$)
- The differential equation that describes this circuit, and
- The solution to $I_a(t)$ and $V_1(t)$.

CircuitLab Solution: One way to find the voltages and currents is to use CircuitLab. One way to do this is to add a switch that opens at $t=0$ as in Figure 2.

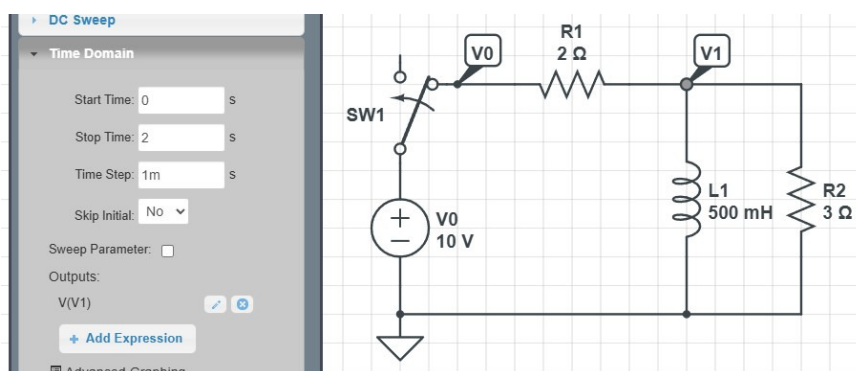


Figure 2: Finding the natural response using CircuitLab

Running a time-domain simulation results in the voltage shown in Figure 3.

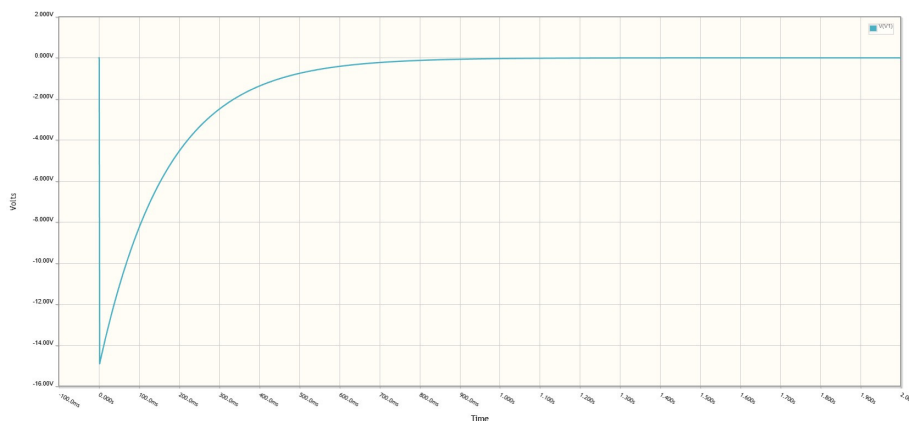


Figure 3: Voltage $V_1(t)$ for $t>0$ as found using CircuitLab

Note that this response seems kind of odd:

- There is jump discontinuity in voltage at $t=0$, and
- The +10V source is temporarily producing -15V.

This is actually correct - odd as it is. Remember that for inductors, current is energy, not voltage. This means that the current through the inductor will be a continuous function. Since voltage isn't energy and is instead related to the change in current, jump discontinuities and amplification in voltage is possible with RL circuits.

That is actually the basis for DC-to-DC inverters and DC-to-DC converters that are covered in electronics and power electronics.

As noted earlier, CircuitLab is very good at showing what $V_1(t)$ looks like. To determine the equation for $V_1(t)$, circuit analysis is needed.

Circuit Analysis

First, find the initial condition for $I_a(t)$. This is equivalent for determining the energy stored in the inductor at $t=0$. Drawing the circuit for $t<0$ results in the circuit of Figure 4.

Since the input is a DC signal, this is a steady-state problem. At DC, inductors have zero ohms or zero volts across them:

$$V = L \frac{dI}{dt} = L \cdot 0$$

This tells you that $V_1(0) = 0$. The current is then

$$I_a(0) = \left(\frac{10V - 0V}{2\Omega} \right) = 5A$$

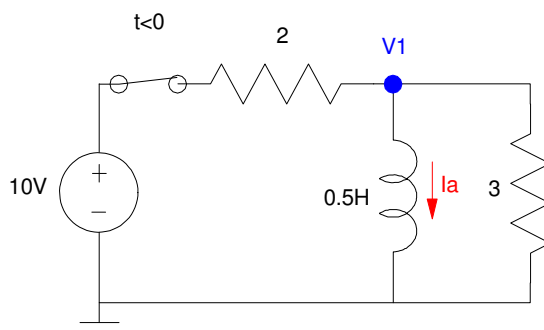


Figure 4: RL circuit for $t < 0$

For $t > 0$, the switch opens, resulting in the circuit of Figure 5. Replacing the inductor with its impedance using the p-operator gives

$$Z_L = Lp$$

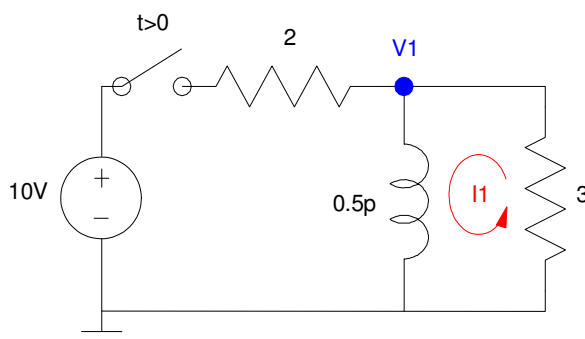


Figure 5: RL circuit for $t > 0$

The differential equation can be found by writing the loop equation around I1

$$(0.5p)I_1 + (3)I_1 = 0$$

Simplifying gives

$$(p + 6)I_1 = 0$$

This tells you that either

$$I_1(t) = 0$$

(the trivial solution), or

$$p = -6$$

Going with the latter, I(t) is of the form

$$I_1(t) = a \cdot e^{-6t}$$

Plugging in the initial condition

$$I_1(0) = 5 = a \cdot e^{0t}$$

$$a = 5$$

giving the natural response for I1(t) for t>0 as shown in Figure 6.

$$I_1(t) = 5 \cdot e^{-6t}$$

Note that I1(t) is a continuous function. This is due to the energy in the system being a function of I1. If the energy in the system is a continuous function, I1(t) must also be a continuous function.

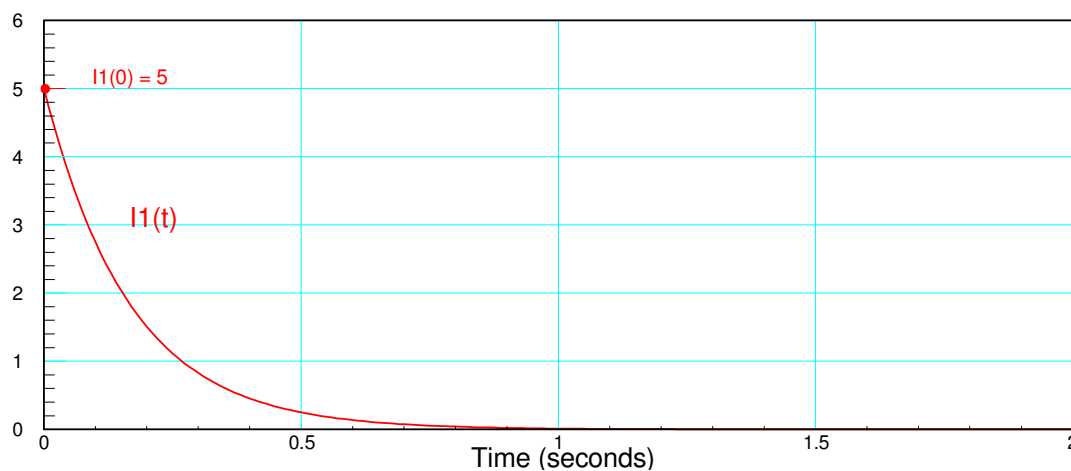


Figure 6: Natural response for I1(t) for a 1-stage RL circuit

Note that the voltage V1(t) is related to the derivative of the current

$$V_1 = L \cdot \frac{dI_1}{dt}$$

The slope of $I_1(t)$ suddenly changes at $t=0$. Hence, the voltage has a jump discontinuity at $t=0$

$$V_1(t) = 0.5 \cdot \frac{d}{dt}(5e^{-6t})$$

$$V_1(t) = -15e^{-6t} \quad t > 0$$

This results in $V_1(t)$ jumping to -15V at $t=0$ then decaying back to 0V as $\exp(-6t)$ as shown in Figure 7.

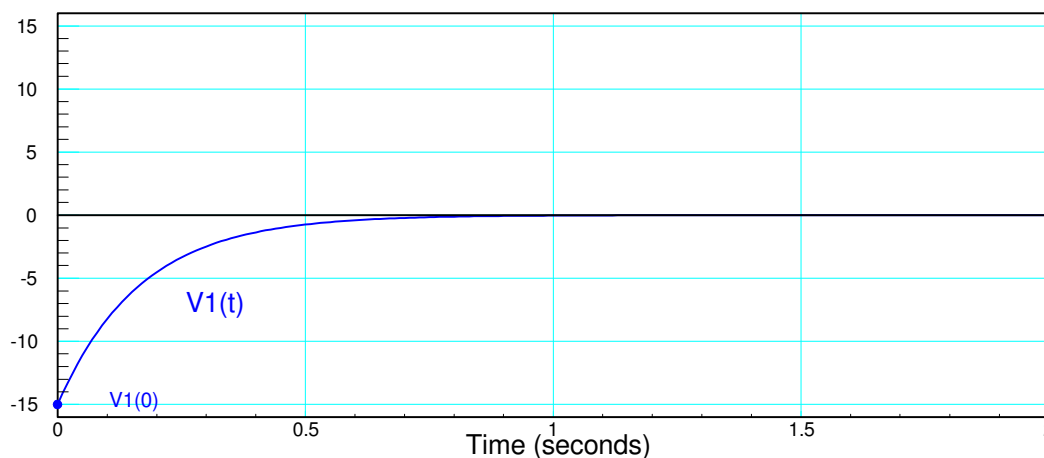


Figure 7: Voltage $V_1(t)$ for $t > 0$ with a 1-stage RL circuit

This voltage inversion and amplification is used in DC-to-DC converters. It is also how spark-plugs work in your car (a 12V battery produces 1000+ volts at the spark plug using an RL circuit.)

2-Stage RL Circuit

Next, let's look at a 2-stage RL circuit as shown in Figure 8. Once again, a switch opens at $t=0$. For this circuit, find

- The initial conditions of $I_1(t)$ and $I_2(t)$ (i.e. find the initial energy in the circuit),
- The differential equation that describes this circuit using the p-operator, and
- Find $V_2(t)$ and $I_2(t)$ for $t > 0$ (i.e. the natural response)

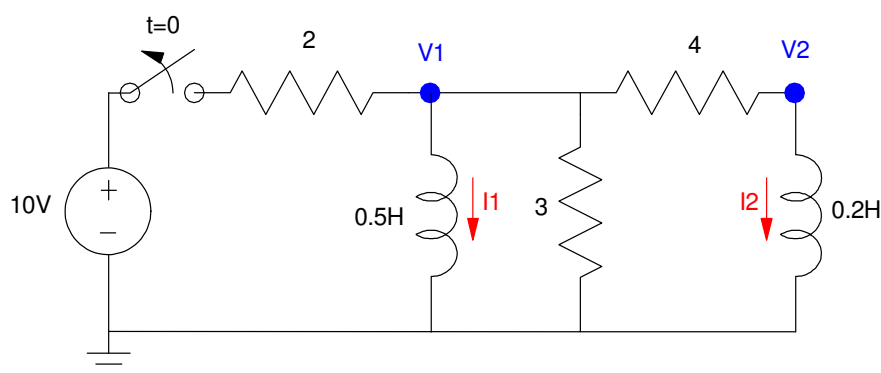


Figure 8: 2-Stage RL circuit with a switch that opens at $t=0$

CircuitLab Solution

One way to find these signals is to use CircuitLab. Start with inputting the circuit with a switch that opens at $t=0$ as shown in Figure 9.

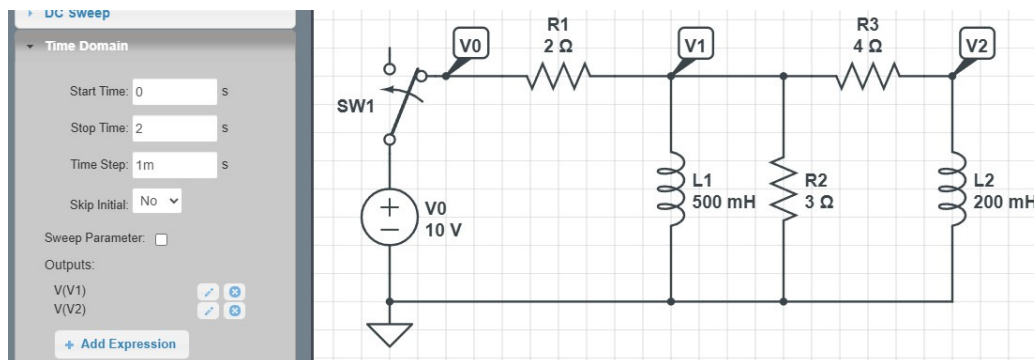


Figure 9: 2-stage RL circuit implementation in CircuitLab

Running a transient simulation gives the voltages $V1(t)$ and $V2(t)$ as shown in Figure 10.

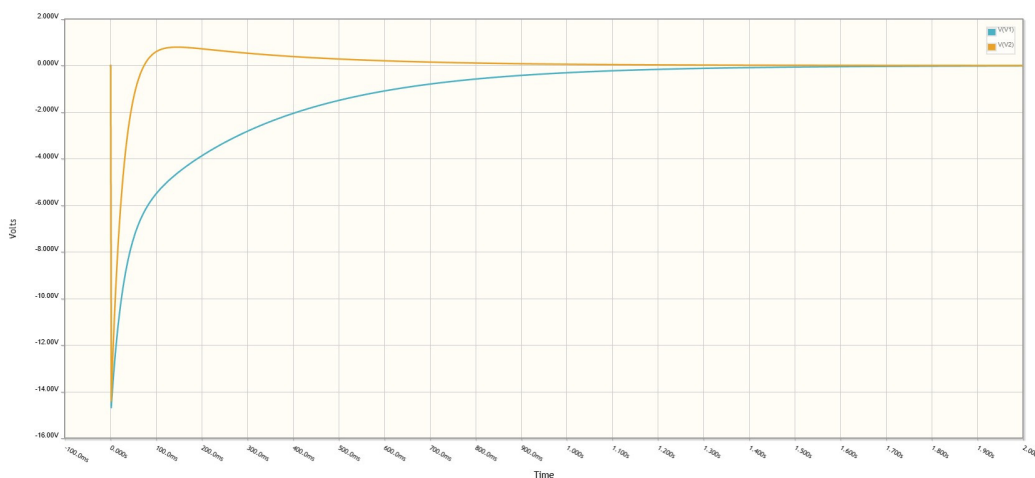
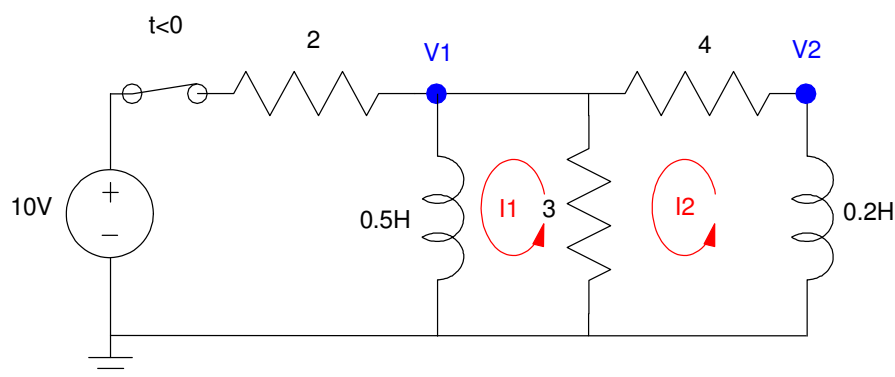


Figure 10: Voltages $V1(t)$ and $V2(t)$ found using CircuitLab for $t>0$

The voltages can look strange for RL circuits. This is OK: voltage isn't energy for inductors. CircuitLab is a very good tool for showing what the resulting voltages look like. To find an equation for these voltages, some analysis is needed.

Circuit Analysis

To analyze the circuit, first find the initial conditions. Redrawing the circuit for $t<0$ results in the circuit of Figure 11.

Figure 11: 2-Stage RL Circuit for $t < 0$

By observation, V_1 and V_2 are the voltages across an inductor for a DC input. This results in

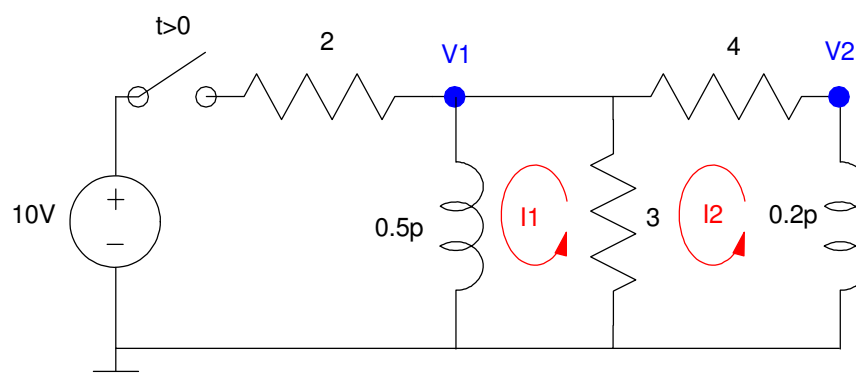
$$V_1 = V_2 = 0$$

This means that

$$I_1(0) = \left(\frac{10V - V_1}{2\Omega} \right) = 5A$$

$$I_2(0) = \left(\frac{V_1 - V_2}{4\Omega} \right) = 0A$$

Now that we know the initial conditions at $t=0$, redraw the circuit for $t>0$ as shown in Figure 12.

Figure 12: 2-Stage RL Circuit for $t > 0$

In Figure 12, the impedance of the inductors is replaced with their impedances using the p -operator. Writing the current-loop equations gives

$$0.5p(I_1) + 3(I_1 - I_2) = 0$$

$$3(I_2 - I_1) + 4(I_2) + 0.2p(I_2) = 0$$

or, with some algebra

$$(0.5p + 3)I_1 - (3)I_2 = 0$$

$$-(3)I_1 + (0.2p + 7)I_2 = 0$$

Solving for I_2 gives

$$(p^2 + 41p + 120)I_2 = 0$$

$$(p + 3.17)(p + 37.12)I_2 = 0$$

This tells you that

$$p = \{-3.17, -37.12\}$$

and that $I_2(t)$ is of the form

$$I_2(t) = ae^{-3.17t} + be^{-37.12t}$$

To find (a, b), plug in the initial conditions.

$$I_1(0) = 5A$$

$$I_2(0) = 0A$$

The former initial condition doesn't really help since the natural response is for $I_2(t)$. Using the differential equations, the derivative of I_2 at $t=0$ can be found

$$-(3)I_1 + (0.2p + 7)I_2 = 0$$

$$pI_2 = 15I_1 - 35I_2$$

or

$$pI_2(0) = I_2'(0) = 75$$

Solving for (a, b) gives

$$I_2(t) = ae^{-3.17t} + be^{-37.83t}$$

$$I_2(0) = 0 = a + b$$

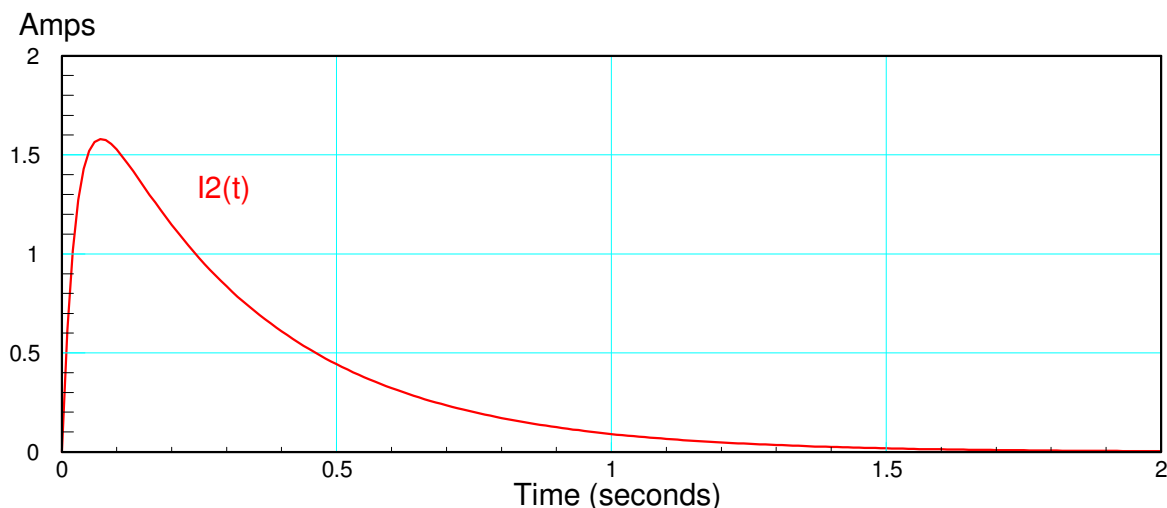
$$I_2'(0) = 75 = -3.17a - 37.12b$$

This gives

- $a = 2.164$
- $b = -2.164$

or

$$I_2(t) = 2.164e^{-3.17t} - 2.164e^{-37.83t}$$

Figure 13: Natural response for $I_2(t)$ for $t > 0$

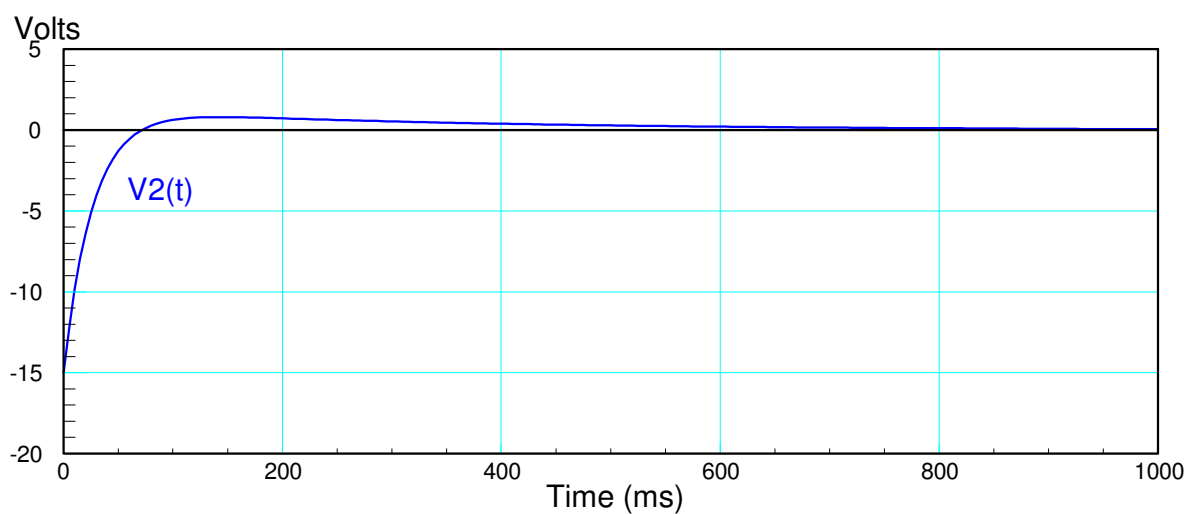
Note that current is a continuous function. For inductors, current is energy. If energy is a continuous function, current will also be a continuous function.

The voltage $V_2(t)$ is related to the derivative of I_2

$$V_2(t) = L \frac{dI_2}{dt}$$

$$V_2(t) = -1.373e^{-3.17t} + 16.373e^{-37.12t}$$

This matches the figure given by CircuitLab

Figure 14: Natural response for $V_2(t)$ for $t > 0$

The voltage $V_2(t)$ may look strange - but again remember for inductors, voltage isn't energy.

Eigenvalues and Eigenvectors

With the previous solution to the 2-stage RL circuit, the initial condition was set by the circuit prior to $t=0$. Suppose you could set the initial condition so that

- $I_2(0) = 1\text{A}$
- $I_1(0) = \text{anything}$

What initial condition on I_1 would result in

- $I_2(t)$ decaying as fast as possible?
- $I_2(t)$ decaying as slow as possible?

How fast or slow is that?

This is an eigenvalue / eigenvector problem. Recall that the dynamics for this circuit are

$$(0.5p + 3)I_1 - (3)I_2 = 0$$

$$-(3)I_1 + (0.2p + 7)I_2 = 0$$

Rewrite this as

$$pI_1 = -6I_1 + 6I_2$$

$$pI_2 = 15I_1 - 35I_2$$

In matrix form

$$\begin{bmatrix} pI_1 \\ pI_2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 15 & -35 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$pI = AI$$

Matrix A is a 2x2 matrix. It likewise has

- Two eigenvalues, and
- Two eigenvectors.

These can be found using Matlab

```
>> A = [-6, 6 ; 15, -35]

      -6      6
      15     -35

>> [M,V] = eig(A);
>> % Eigenvectors
>> M

      0.9046    -0.1853
      0.4263     0.9827

>> % Eigenvalues
>> diag(V) '

     -3.1723    -37.8277
```

Note that the eigenvalues are the exponentials found for $I_2(t)$. Eigenvalues tell you *how* the system will respond. For this circuit, $I_2(t)$ will be of the form

$$I_2(t) = ae^{-3.17t} + be^{-37.82t}$$

Eigenvectors tell you *what* behaves that way. In general, $I(t)$ will be of the form

$$I(t) = \alpha \begin{bmatrix} 0.9046 \\ 0.4263 \end{bmatrix} e^{-3.1723t} + \beta \begin{bmatrix} -0.1853 \\ 0.9827 \end{bmatrix} e^{-37.8277t}$$

The initial condition tells you the value of α and β

Fast Mode: If you make the initial condition proportional to the fast eigenvector, then only the fast mode will be excited. In Matlab

```
>> IC = M(:,2);
>> IC = IC / IC(2)

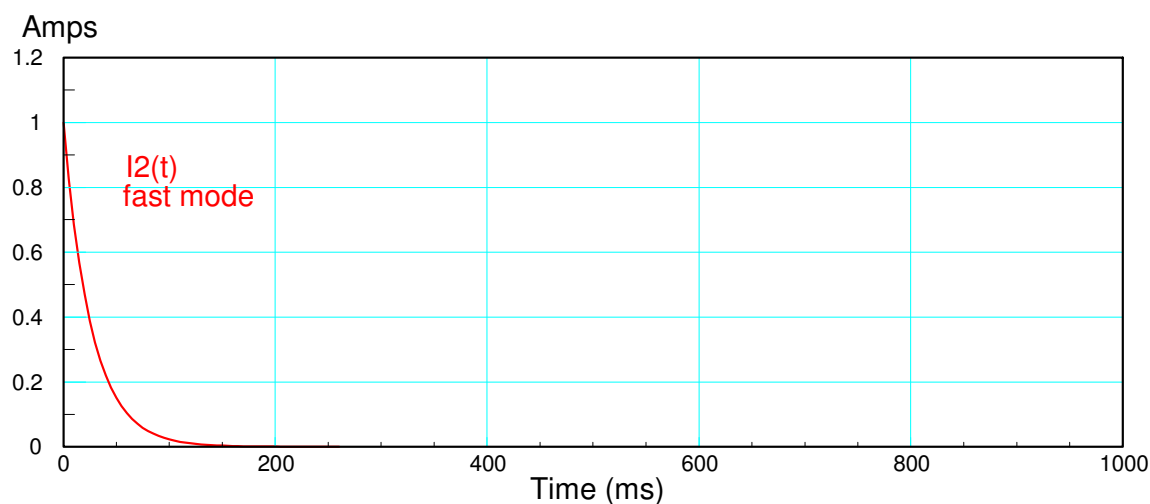
I1(0)    -0.1885
I2(0)     1.0000
```

To excite the fast mode, set

- $I_1(0) = -0.1885\text{A}$ and
- $I_2(0) = 1\text{A}$

This results in

$$I(t) = \begin{bmatrix} -0.1885 \\ 1.000 \end{bmatrix} e^{-37.8277t}$$

Figure 15: $I_2(t)$ when only the fast mode is excited

Slow Mode: If instead, the initial condition is proportional to the slow eigenvector, only that mode is excited

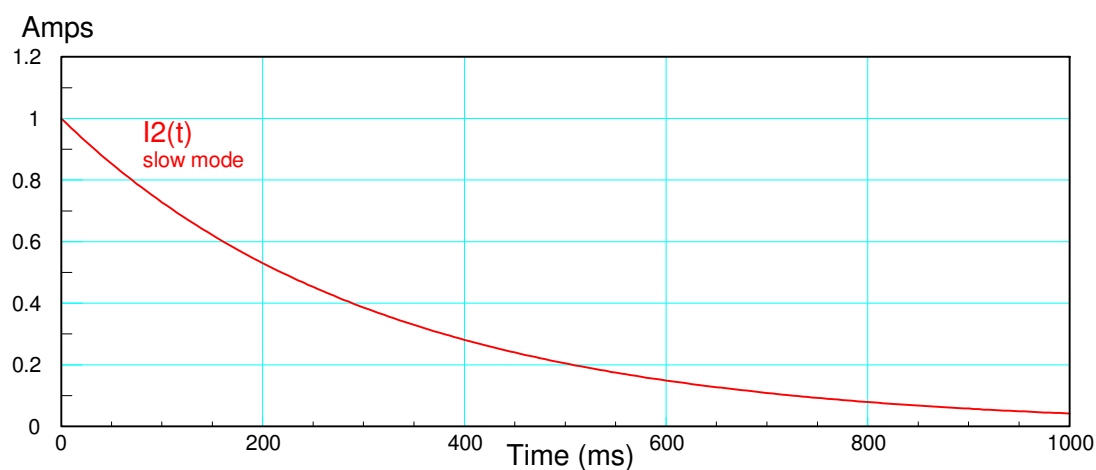
```
>> IC = M(:,1);
>> IC = IC / IC(2)

I1(0)    2.1218
I2(0)    1.0000
```

This means that if you set $I_1(0)$ to 2.1281 Amps, the current will be

$$I(t) = \begin{bmatrix} 2.1218 \\ 1.0000 \end{bmatrix} e^{-3.1723t}$$

and only the slow mode will be excited

Figure 16: Setting $I_1(0) = 2.1218\text{A}$ excites only the slow mode

If you have some other initial condition, such as

$$I_1(0) = I_2(0) = 1\text{A}$$

then both modes are excited. In Matlab, the initial condition can using Matlab:

```
>> A = [-6, 6 ; 15, -35]

      -6      6
      15     -35

>> [M,V] = eig(A)

M =
    0.9046   -0.1853
    0.4263    0.9827

V =
   -3.1723         0
         0   -37.8277

>> IC = inv(M) * [1;1]

a =      1.2067
b =      0.4941
```

This means that

$$I(t) = a \begin{bmatrix} 0.9046 \\ 0.4263 \end{bmatrix} e^{-3.1723t} + b \begin{bmatrix} -0.1853 \\ 0.9827 \end{bmatrix} e^{-37.8277t}$$

with

- $a = 1.2067$
- $b = 0.4941$

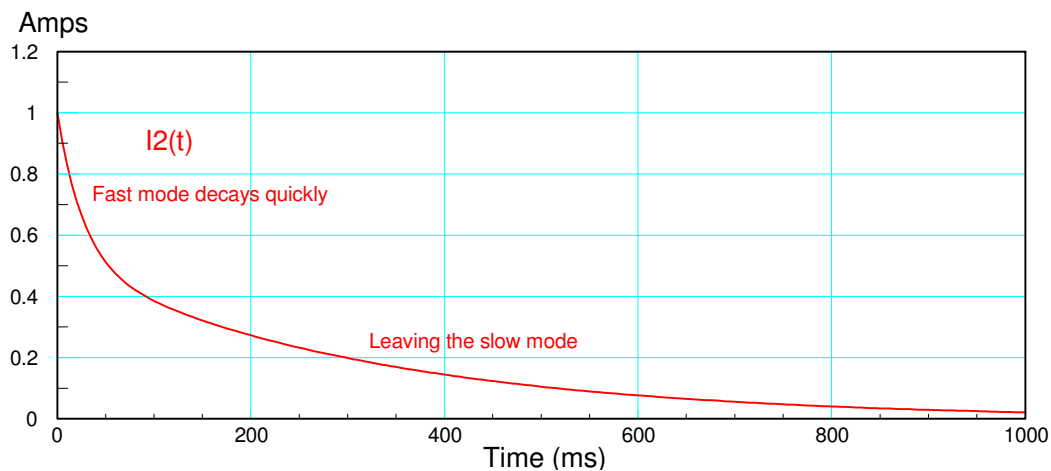


Figure 17: Initial conditions of $I_1(0) = I_2(0) = 1\text{A}$ excite both the fast mode and the slow mode

Note that in Figure 17, you can see both the fast and the slow mode.

- Initially, the fast mode is excited. It quickly decays
- Leaving just the slow mode.

After a short while, all that's left is the slow mode. This mode tends to dominate the response for $I_2(t)$ - and hence the slow mode is also called *the dominant mode*.

Summary

The natural response of an RL circuit is the response when

- The inductors have an initial current and
- The input turns off.

The natural response can be found using

- Using CircuitLab,
- The differential equation that describe the circuit, and
- With the circuit's eigenvalues and eigenvectors.