

Natural Response of RC Circuits

Background

Given an RLC circuit with an input that turns on at $t=0$, the voltages will be the sum of two terms: the forced response and the natural response. The forced response is what we've been finding so far. Using phasors, the steady-state solution can be found. This is the forced response. The natural response is the transient that starts from the initial conditions at $t=0$ and leads up to the steady-state solution.

In this lecture, the natural response for a 1-stage and 2-stage RC filter similar to Figure 1 will be found. In both cases, a power supply provides an initial condition on the voltage (i.e. energy) in each capacitor. At $t=0$, the power supply is removed from the circuit. This results in the forced response being $V_1=0$. The natural response is then the path the voltages take to go from their initial condition to the steady-state solution: $V=0$.

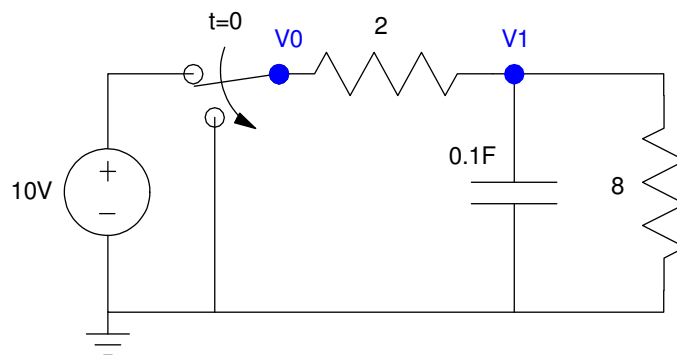


Figure 1: Find the natural response of a 1-stage RC filter

1-State RC Circuit

Consider first the 1-stage RC circuit of Figure 1. Assume the switch changes at $t=0$. Find $V_1(t)$ for $t>0$.

CircuitLab Solution: One way to solve this problem is to throw it into CircuitLab. One such solution is to use a switch as shown in Figure 2.

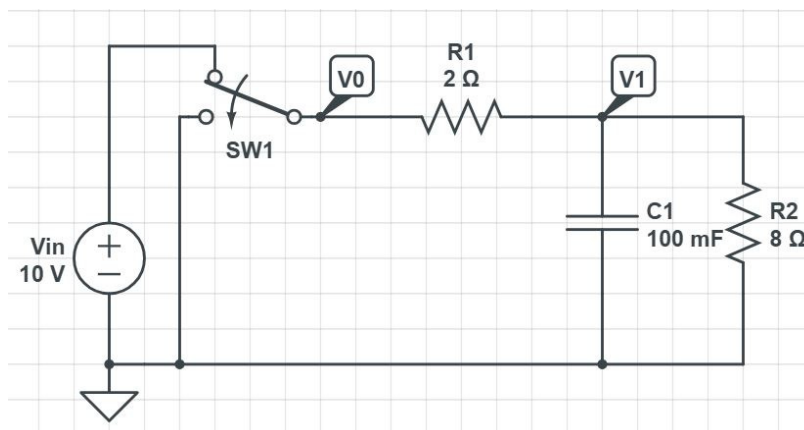


Figure 2: CircuitLab simulation using a switch that changes at $t=0$.

Figure 3 shows the voltage $V_1(t)$ found by using a time-domain simulation. From the simulation, it can be observed that

- $V_1(t)$ starts at 8.00V
- It decays down to 0.00V
- In about 700ns

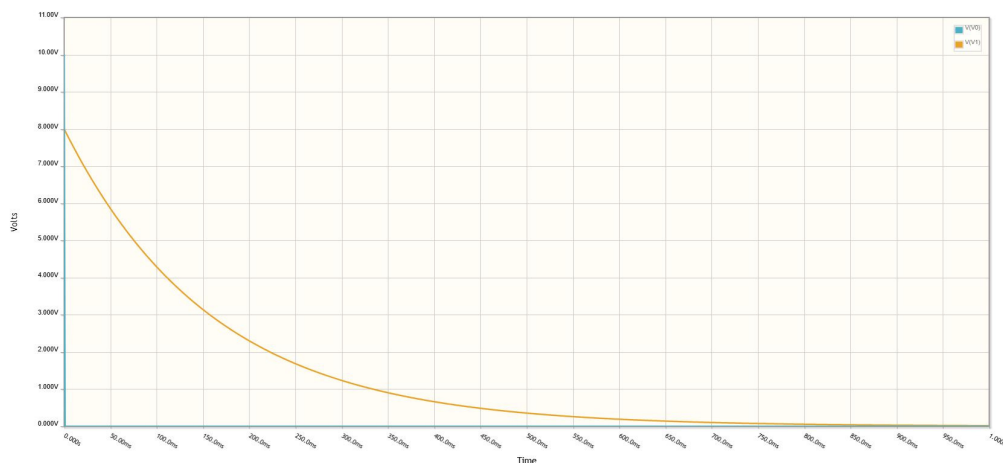


Figure 3: $V_1(t)$ (natural response) found using CircuitLab

Another way to find the natural response is replace V_0 with a square wave that goes from 0V to 5V as shown in Figure 4.

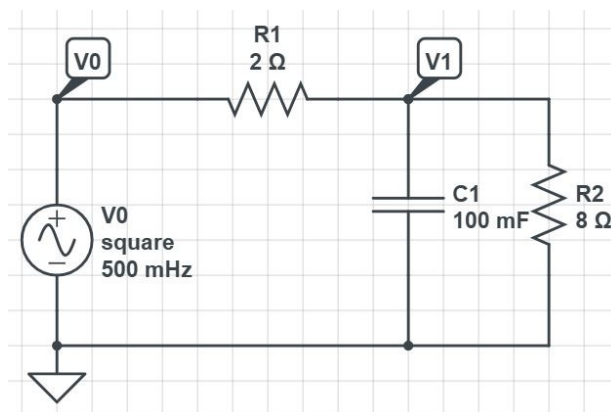


Figure 4: Another way to find the natural response: Make V_0 a 0V / 5V square wave with a period that long enough for the transients to die out.

A time-domain simulation of this circuit gives two responses:

- When $V_0(t) = 5V$, $V_1(t)$ shows the forced response ($V_1(t) = 4V$) and the natural response
- When $V_0(t) = 0V$, $V_1(t)$ just shows the natural response (the forced response is $V_1(t) = 0$).

This is shown in Figure 5.

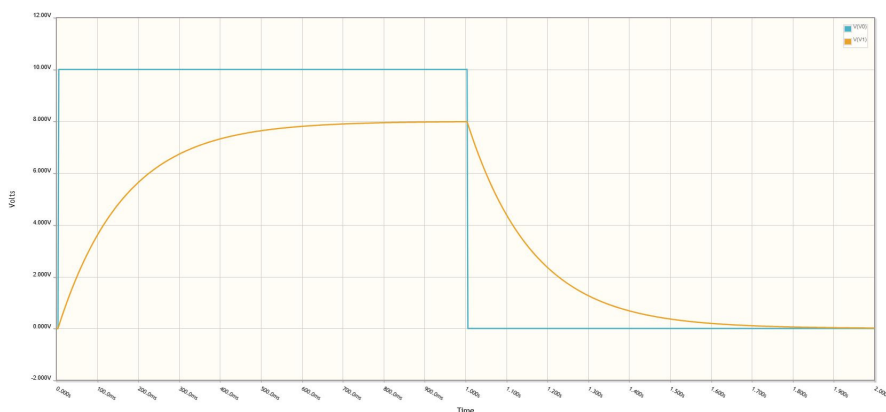


Figure 5: $V_0(t)$ (blue) and $V_1(t)$ (orange) for a 0V / 5V square wave input.

Note that in lab, the square wave is an easier way to find $V_1(t)$. Oscilloscopes like periodic inputs so that the trace can be updated over and over again. You just have to make the period of the square wave long enough so that the transients have time to decay before V_0 switches.

Analytic Solution: CircuitLab is a pretty nice tool: it shows you what $V_1(t)$ should look like for $t > 0$. One problem with CircuitLab is it doesn't tell you the equation for $V_1(t)$. To find that, you need to do some analysis.

Start with using the p-operator to represent the impedance of a capacitor. The VI relationship for a capacitor is

$$I_1 = C \frac{dV_1}{dt} = CpV_1$$

That makes the impedance of a 0.1F capacitor

$$Z_c = \frac{V_1}{I_1} = \frac{1}{Cp} = \frac{10}{p}$$

The voltage node equation at V_1 is

$$\left(\frac{V_1 - 0}{2} \right) + \left(\frac{V_1}{10/p} \right) + \left(\frac{V_1}{8} \right) = 0$$

which simplifies to

$$(p + 6.25)V_1 = 0$$

If $V_1(t)$ isn't zero, then

$$p = -6.25$$

This tells you that $V_1(t)$ is of the form

$$V_1(t) = a \cdot e^{pt}$$

$$V_1(t) = a \cdot e^{-6.25t}$$

Just prior to $t=0$, $V_0=10$. By voltage division, this makes the initial condition on V_1 to be

$$V_1(0) = \left(\frac{8}{8+2}\right) 10$$

$$V_1(0) = 8V$$

This tells you that $a=8$ and $V_1(t)$ is

$$V_1(t) = 8e^{-6.25t} \quad \text{for } t>0$$

This is the equation for $V_1(t)$ that CircuitLab computed using numerical methods.

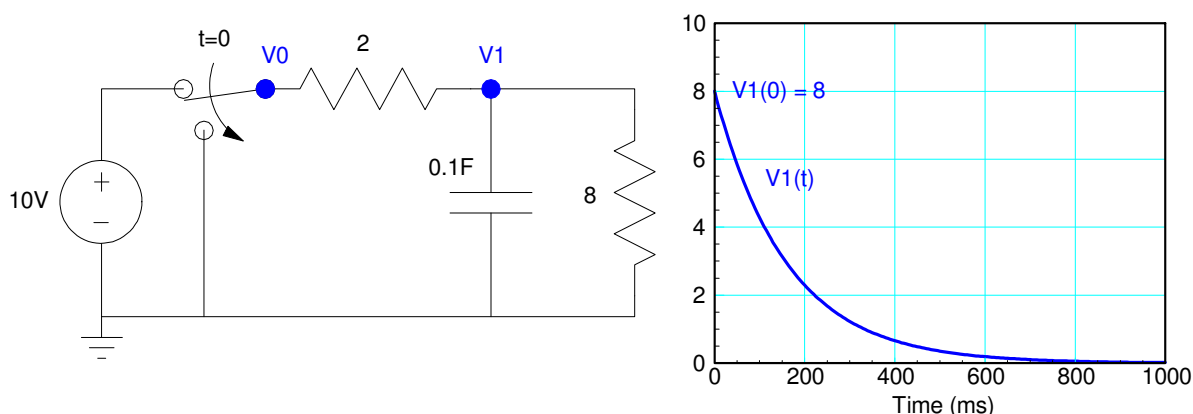


Figure 6: $V_1(t)$ for $t>0$

Natural Response for a 2-Stage RC Circuit

Consider next a 2-stage RC circuit as shown in Figure 7. Once again, the switch connected the circuit to a 10V supply prior to $t=0$. This sets an initial voltage (and energy) on the two capacitors. At $t=0$, the switch changes. This removes the source, resulting in the forced response for $V_1(t)$ and $V_2(t)$ being 0V. The time response for $V_1(t)$ and $V_2(t)$ for $t>0$ are thus the natural response of these voltages.

How do you find $V_1(t)$ and $V_2(t)$?

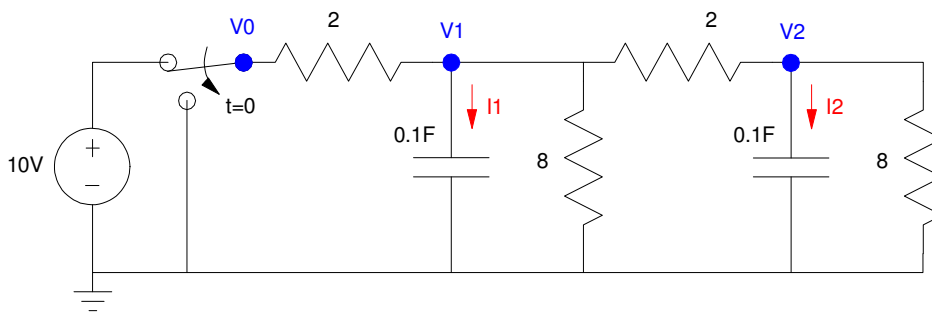


Figure 7: 2-Stage RC Circuit. $V_1(t)$ and $V_2(t)$ for $t>0$ are the natural response

CircuitLab Solution: One way to find these voltages is to throw the circuit into CircuitLab and run a time-domain simulation. One way to do this is to use a switch that changes at $t=0$, as shown in Figure 8.

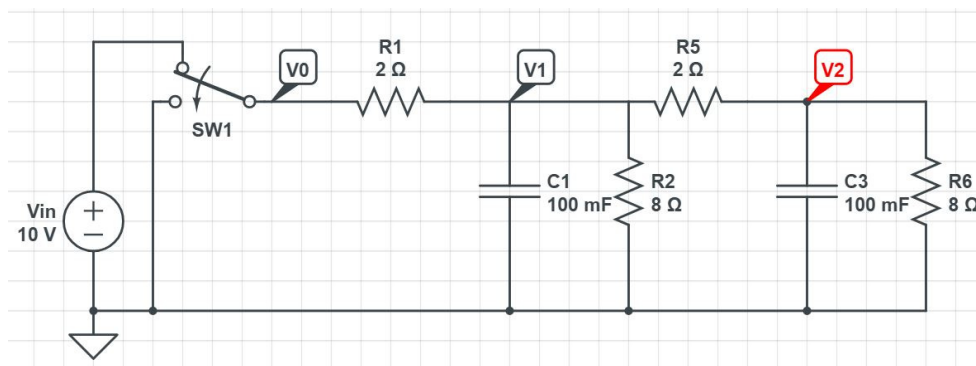


Figure 8: CircuitLab circuit which uses a switch that changes at $t=0$

Running a time-domain simulation gives $V1(t)$ and $V2(t)$ as shown in Figure 9.

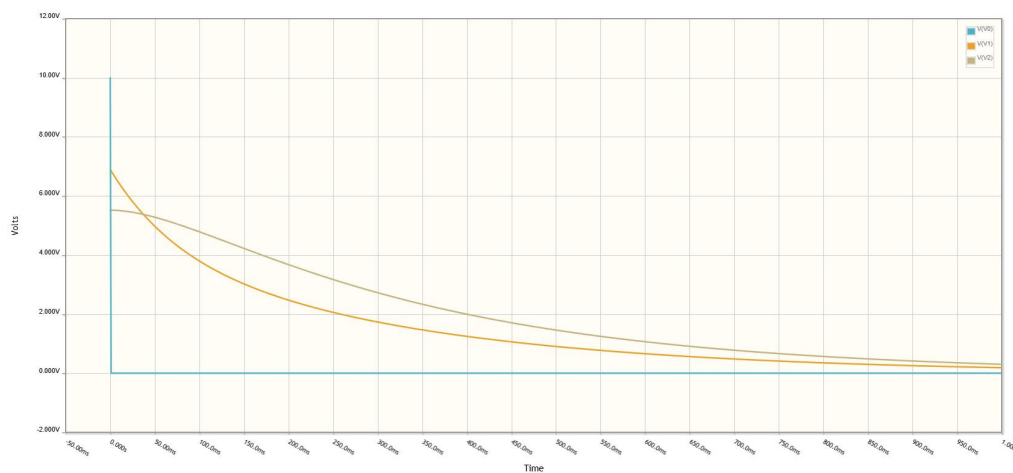


Figure 9: $V0(t)$ (blue), $V1(t)$ (orange), and $V2(t)$ (tan) found using CircuitLab

You could also use a 0V / 5V square wave for $V0$ as shown in Figure 9.

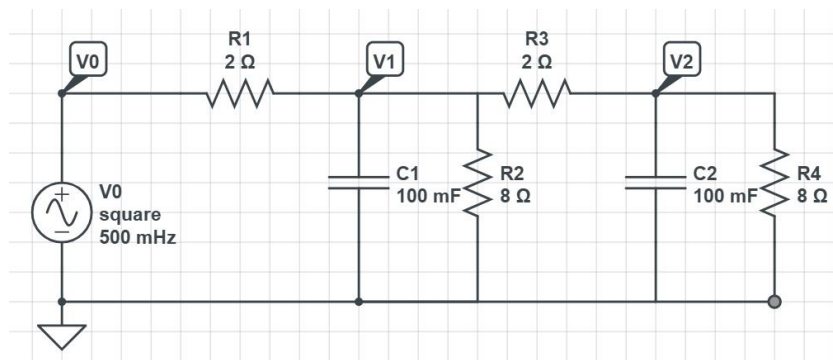


Figure 9: Another way to find $V_1(t)$ and $V_2(t)$ is to use a square wave input

As long as the period of V_0 is long enough that V_1 and V_2 reach steady-state before switching, this circuit also gives the natural response (when $V_0=0$) as well as a forced + natural response (when $V_0 = 5V$).

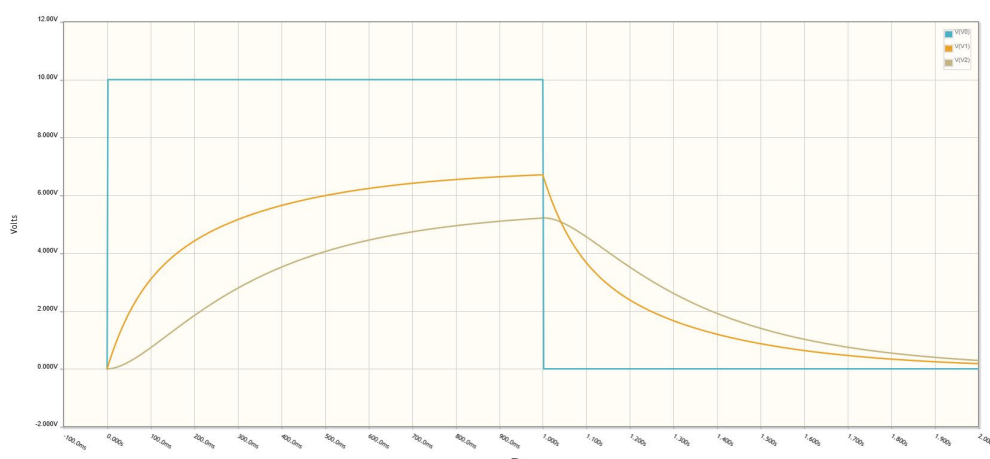


Figure 10: A time-domain simulation gives the natural response when $V_0(t) = 0$

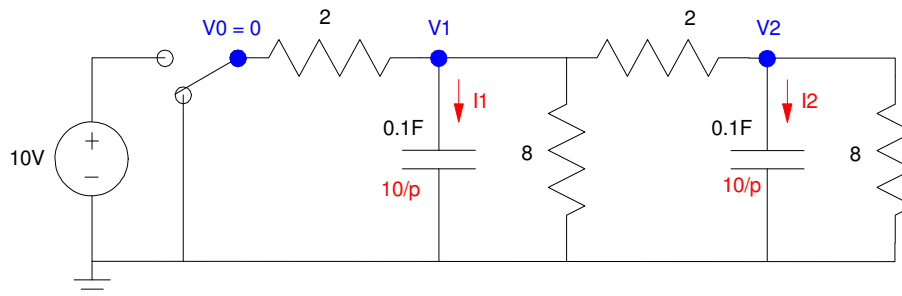
Once again, CircuitLab tells you what $V_1(t)$ and $V_2(t)$ look like. To find an equation for these curves, some analysis is required.

Circuit Analysis

Like before, replace the capacitors with their impedance using the p-operator

$$Z_c = \frac{1}{Cp} = \frac{10}{p}$$

This results in the circuit for $t > 0$ being as shown in Figure 11.

Figure 11: Circuit for $t > 0$

Writing the voltage node equations:

$$\left(\frac{V_1}{2}\right) + \left(\frac{V_1}{10/p}\right) + \left(\frac{V_1}{8}\right) + \left(\frac{V_1 - V_2}{2}\right) = 0$$

$$\left(\frac{V_2 - V_1}{2}\right) + \left(\frac{V_2}{10/p}\right) + \left(\frac{V_2}{8}\right) = 0$$

Doing some algebra

$$(p + 11.25)V_1 - (5)V_2 = 0$$

$$-(5)V_1 + (p + 6.25)V_2 = 0$$

Some more algebra lets you solve for V_2

$$(p^2 + 17.5p + 45.3125)V_2 = 0$$

$$(p + 14.3402)(p + 3.1598)V_2 = 0$$

This tells you that, if $V_2(t)$ is not zero, then

$$p = \{-14.3402, -3.1598\}$$

and the natural response for $V_2(t)$ is of the form

$$V_2(t) = ae^{-14.3402t} + be^{-3.1598t}$$

To find 'a' and 'b', use the initial conditions.

Prior to $t=0$, the initial voltages are

- $V_0 = 10.000$
- $V_1 = 6.897$
- $V_2 = 5.517$

meaning

$$V_2(0) = 5.517$$

To find a second equation for 'a' and 'b', use the differential equation found previously

$$-(5)V_1 + (p + 6.25)V_2 = 0$$

This tells you what $V_2'(0)$ is

$$pV_2 = 5V_1 - 6.25V_2$$

$$pV_2(0) = V_2'(0) = 0$$

Plug these into the general form for $V_2(t)$

$$V_2(t) = ae^{-14.3402t} + be^{-3.1598t}$$

$$V_2(0) = 5.517 = a + b$$

$$V_2'(0) = 0 = -14.3402a - 3.1598b$$

Solving (doing some algebra)

$$a = -1.5592$$

$$b = 7.0762$$

and

$$V_2(t) = -1.5592e^{-14.3402t} + 7.0762e^{-3.1598t}$$

This is the equation for $V_2(t)$ given by CircuitLab

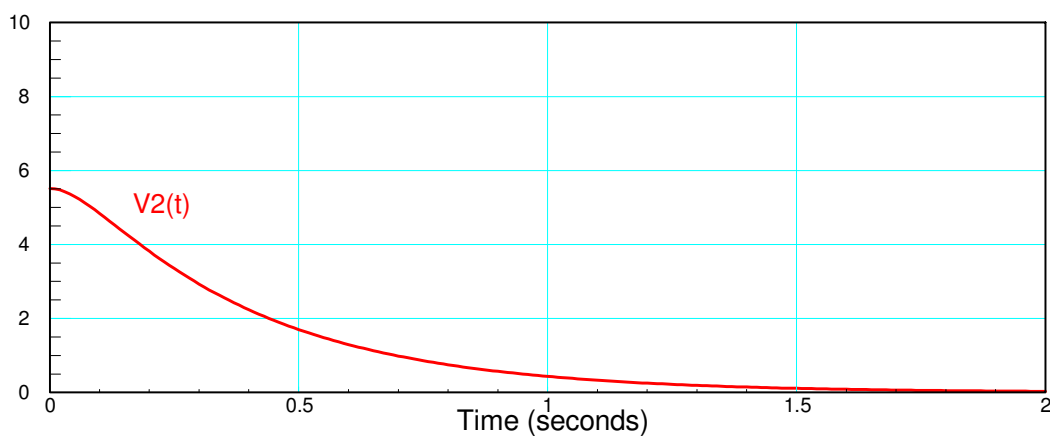


Figure 12: $V_2(t)$

Eigenvalues & Eigenvectors

The previous solution is $V_2(t)$ given the initial conditions

- $V_1(0) = 6.897V$
- $V_2(0) = 5.517V$

This leads to another question.

Suppose $V_2(0) = 10V$. What initial condition on $V_1(t)$ results in

- $V_2(t)$ decaying as fast as possible?
- $V_2(t)$ decaying as slow as possible?
- How fast or slow is that?

Also, what happens when $V_1(0) = 10V$?

This is an eigenvalue / eigenvector problem.

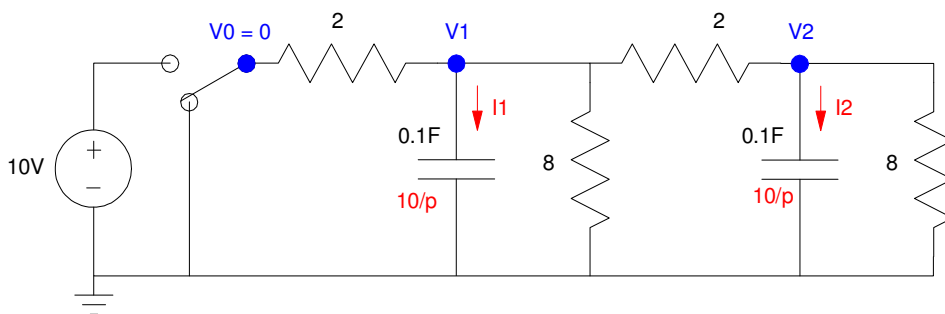


Figure 13: Find the initial condition on $V_1(t)$ to make $V_2(t)$ decay quickly or slowly.

To see how this works, let's go back to the differential equations for this circuit:

$$(p + 11.25)V_1 - (5)V_2 = 0$$

$$-(5)V_1 + (p + 6.25)V_2 = 0$$

Solving each for the derivatives

$$pV_1 = -11.25V_1 + 5V_2$$

$$pV_2 = 5V_1 - 6.25V_2$$

Place in matrix form

$$\begin{bmatrix} pV_1 \\ pV_2 \end{bmatrix} = \begin{bmatrix} -11.25 & 5 \\ 5 & -6.25 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$pV = AV$$

Matrix A is a 2x2 matrix. From linear algebra,

- A has two eigenvalues
- A has two eigenvectors.

In Matlab, these can be found as follows:

```
>> A = [-11.25, 5 ; 5, -6.25]

    -11.2500    5.0000
     5.0000   -6.2500

>> [M,N] = eig(A)

eigenvectors
V1    -0.8507   -0.5257
V2     0.5257   -0.8507

eigenvalues
    -14.3402   -3.1598
```

Eigenvalues tell you *how* the systems behaves:

- Something decays as $\exp(-14.3402t)$
- Something else decays as $\exp(-3.1598t)$

Note that we already knew that. This is the form of the natural response

$$V_2(t) = ae^{-14.3402t} + be^{-3.1598t}$$

Eigenvectors tell you *what* behaves that way. In general, the natural response will be of the form

$$V(t) = \alpha \cdot \begin{bmatrix} -0.8507 \\ 0.5257 \end{bmatrix} \cdot e^{-14.3402t} + \beta \cdot \begin{bmatrix} -0.5257 \\ -0.8507 \end{bmatrix} \cdot e^{-3.1598t}$$

To make the transient decay quickly, use the first eigenvector. In Matlab:

```
>> V0 = M(:,1)

    -0.8507
     0.5257

>> V0 = V0 / V0(2)

V1    -1.6180
V2     1.0000
```

Make $V_1(0) = -1.6180V$ and $V_2(t)$ will decay as $\exp(-14.3402t)$

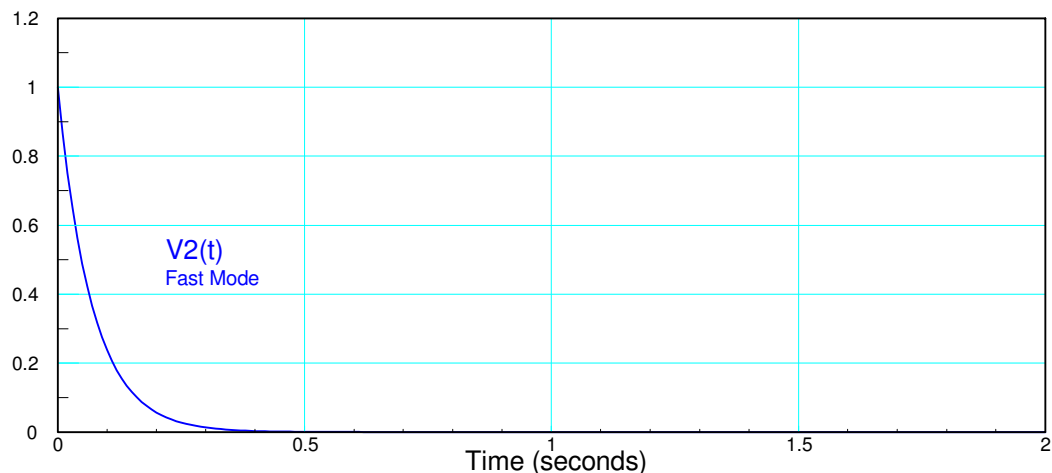


Figure 14: Setting $V_1(0) = -1.618V$, $V_2(0) = 1$ excites the fast mode (the fast eigenvector)

To make the transient decay slowly, use the second eigenvector

```
>> V0 = M(:,2)
      -0.5257
      -0.8507

>> V0 = V0 / V0(2)

V1    0.6180
V2    1.0000
```

If $V_1(0) = 0.6180V$, then $V_2(t)$ will decay as $\exp(-3.1598t)$

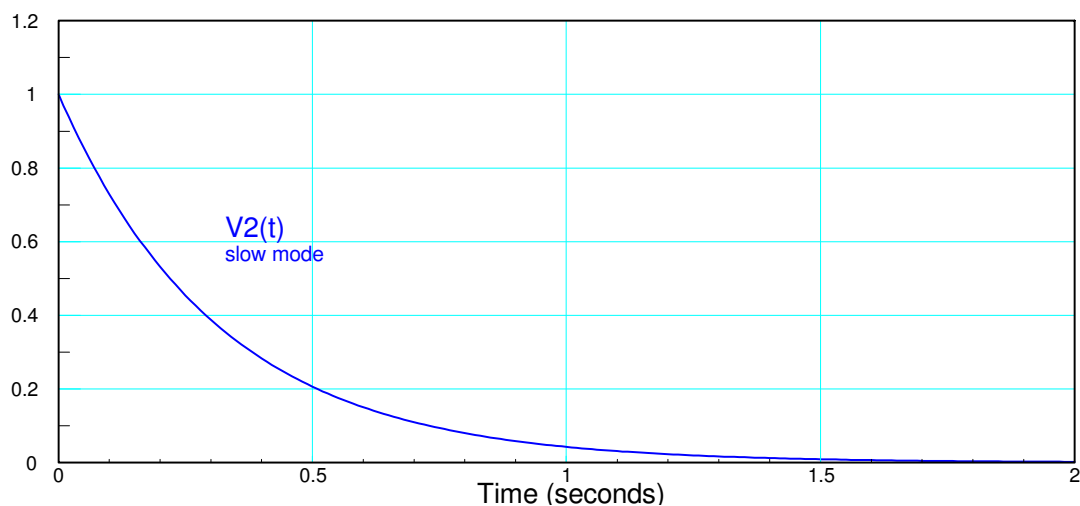


Figure 16: Setting $V_1(0) = 0.6180V$ and $V_2(0) = 1V$ excites the slow eigenvector

Finally, a general initial condition such as

$$V_1(0) = V_2(0) = 1V$$

usually excites both eigenvectors. In Matlab

```
>> IC = [1;1]

      1
      1

>> inv(M) * IC

a   -0.3249
b   -1.3764
```

meaning

$$V_2(t) = a \cdot 0.5257 \cdot e^{-14.34t} + b \cdot (-0.8507) \cdot e^{-3.16t}$$

In this case, $V_2(t)$ is dominated by the slow mode (and hence it is called *the dominant mode*). Initially, the fast mode is present. It quickly decays, however, leaving just the slow mode. A different initial condition (closer to the fast eigenvector) would make the fast mode easier to see.

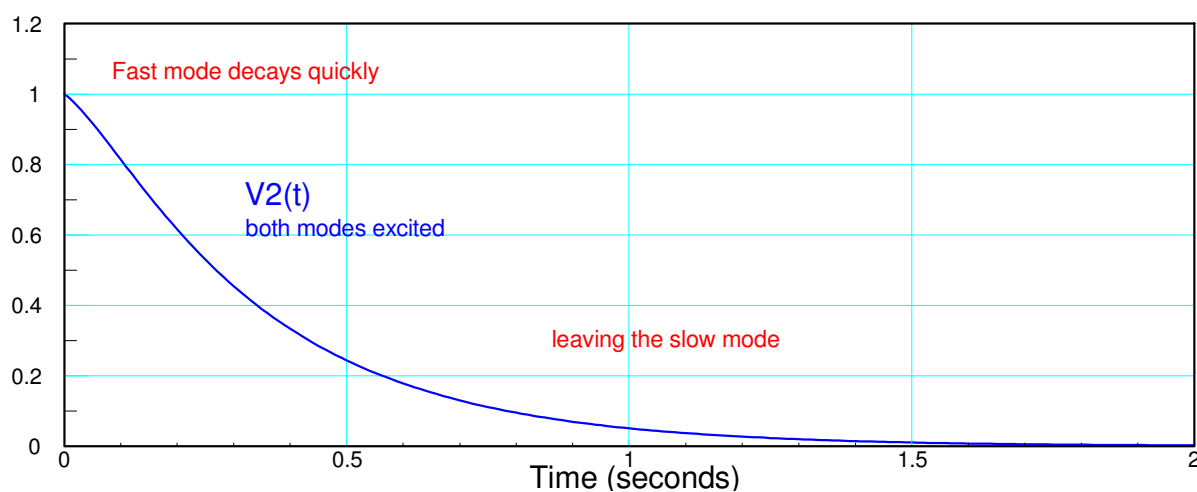


Figure 17: An initial condition of $V_1(0) = V_2(0) = 1$ excites both modes

Summary

The natural response of an RC filter is the response when the capacitors have an initial voltage and the input turns off. The natural response can be found several ways:

- Running a time-domain simulation in CircuitLab, or
- Solving the differential equations that describe the circuit.

The p-operator is useful for the latter.

For a circuit with N capacitors, there will in general be N differential equations that describe this circuit. These N equations will have N eigenvalues and N eigenvectors.

- The eigenvalues tell you *how* the circuit behaves. These are the exponentials in the equations for $V(t)$.
- The eigenvectors tell you *what* behaves that way.

If the initial conditions are proportional to a given eigenvector, then only that eigenvector is excited. A generic initial condition usually excites all of the eigenvectors.